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Feedback boundary stabilization of the three-dimensional incompressible Navier–Stokes equations

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Abstract

We study the local stabilization of the three-dimensional Navier–Stokes equations around an unstable stationary solution **w**, by means of a feedback boundary control. We first determine a feedback law for the linearized system around **w**. Next, we show that this feedback provides a local stabilization of the Navier–Stokes equations. To deal with the nonlinear term, the solutions to the closed loop system must be in $H^{3/2+\varepsilon,3/4+\varepsilon/2}(Q)$, with $0<\varepsilon$. In [V. Barbu, I. Lasiecka, R. Triggiani, Boundary stabilization of Navier–Stokes equations, Mem. Amer. Math. Soc. 852 (2006); V. Barbu, I. Lasiecka, R. Triggiani, Abstract settings for tangential boundary stabilization of Navier–Stokes equations by high- and low-gain feedback controllers, Nonlinear Anal. 64 (2006) 2704–2746], such a regularity is achieved with a feedback obtained by minimizing a functional involving a norm of the state variable strong enough. In that case, the feedback controller cannot be determined by a well posed Riccati equation. Here, we choose a functional involving a very weak norm of the state variable. The compatibility condition between the initial state and the feedback controller at $t=0$, is achieved by choosing a time varying control operator in a neighbourhood of $t=0$.

Résumé

Nous étudions la stabilisation locale des équations de Navier–Stokes en 3D au voisinage d'une solution stationnaire instable **w**, par un contrôle feedback frontière. Nous déterminons d'abord une loi de feedback pour le système linéarisé en **w**. Nous montrons que cette loi donne une stabilisation locale des équations de Navier–Stokes. Pour traiter le terme non linéaire, les solutions du système en boucle fermée doivent appartenir à $H^{3/2+\varepsilon,3/4+\varepsilon/2}(Q)$, avec $0<\varepsilon$. Dans [V. Barbu, I. Lasiecka, R. Triggiani, Boundary stabilization of Navier–Stokes equations, Mem. Amer. Math. Soc. 852 (2006) ; V. Barbu, I. Lasiecka, R. Triggiani, Abstract settings for tangential boundary stabilization of Navier–Stokes equations by high- and low-gain feedback controllers, Nonlinear Anal. 64 (2006) 2704–2746], cette régularité est obtenue avec un feedback déterminé par minimisation d'une fonctionnelle contenant une norme de la variable d'état suffisamment forte. Dans ce cas, le feedback ne peut pas être caractérisé par une équation de Riccati bien posée. Ici, nous choisissons une fonctionnelle contenant une norme très faible de la variable d'état. La condition de compatibilité entre la condition initiale et la loi de contrôle en $t=0$ est obtenue en choisissant un opérateur de contrôle dépendant du temps dans un voisinage de $t=0$.

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1. Introduction

An important issue in control theory is the controllability of systems. For the linearized three-dimensional Navier–Stokes equations the controllability to trajectories has been addressed in [12] (see also the references therein and [9] for earlier results). The local stabilizability of the three-dimensional Navier–Stokes equations in a neighbourhood of an unstable stationary solution may be deduced from this controllability result. Another important issue is the characterization of stabilizing feedback control laws, pointwise in time. In the case of the Navier–Stokes equations, this question has been studied in [4] for distributed controls. For boundary controls, feedback controls are characterized in [13,14], but the corresponding laws are not pointwise in time. In the two-dimensional case [19], we have obtained boundary feedback control laws, pointwise in time, by considering an optimal control problem in which the observation operator is the identity in the velocity space endowed with the $\mathbf{L}^2$-norm. The three-dimensional case is more delicate (see [5,6]). In [5], the existence of boundary feedback laws, pointwise in time, has been established by solving a control problem with a cost functional involving the $\mathbf{H}^{3/2+\varepsilon}$-norm of the velocity field, for some $\varepsilon > 0$ small enough. But, as explained in [6] (see also farther in the introduction), such a feedback law cannot be characterized by a well posed Riccati equation. This is a serious drawback if we want to calculate such feedback control laws by using a numerical approximation of the Riccati equation. The main objective of this paper is to determine a feedback boundary control law, pointwise in time, characterized by a well posed Riccati equation, for the three-dimensional Navier–Stokes equations.

More precisely, let Ω be a bounded and connected domain in $\mathbb{R}^3$ with a regular boundary Γ, $\nu > 0$, and consider a pair $(\mathbf{w}, \chi)$ solution to the stationary Navier–Stokes equations in Ω:

$$-\nu\Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla\chi = \mathbf{f} \quad \text{and} \quad \operatorname{div}\mathbf{w} = 0 \text{ in } \Omega, \quad \mathbf{w} = \mathbf{u}_s^\infty \text{ on } \Gamma.$$

We assume that $\mathbf{w}$ is regular and is an unstable solution of the instationary Navier–Stokes equations. We want to determine a Dirichlet boundary control $\mathbf{u}$, in feedback form, localized in a part of the boundary Γ, so that the corresponding controlled system:

$$\begin{aligned}
&\frac{\partial\mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = 0, \quad \operatorname{div}\mathbf{y} = 0 \quad \text{in } Q_\infty,\\
&\mathbf{y} = M\mathbf{u} \text{ on } \Sigma_\infty, \quad \mathbf{y}(0) = \mathbf{y}_0 \text{ in } \Omega,
\end{aligned} \tag{1.1}$$

be stable for initial values $\mathbf{y}_0$ small enough in an appropriate space $\mathbf{X}(\Omega)$. In this setting, $Q_\infty = \Omega\times(0,\infty)$, $\Sigma_\infty = \Gamma\times(0,\infty)$, $\mathbf{X}(\Omega)$ is a subspace of $\mathbf{V}_n^0(\Omega) = \{\mathbf{y}\in\mathbf{L}^2(\Omega) \mid \operatorname{div}\mathbf{y} = 0 \text{ in } \Omega,\ \mathbf{y}\cdot\mathbf{n} = 0 \text{ on } \Gamma\}$, and the operator M is a restriction operator precisely defined in Section 2. If we set $(\mathbf{z}, q) = (\mathbf{w}+\mathbf{y}, \chi + p)$ and if $\mathbf{u} = 0$, we see that $(\mathbf{z}, q)$ is the solution to the Navier–Stokes equations

$$\begin{aligned}
&\frac{\partial\mathbf{z}}{\partial t} - \nu\Delta\mathbf{z} + (\mathbf{z}\cdot\nabla)\mathbf{z} + \nabla q = \mathbf{f}, \quad \operatorname{div}\mathbf{z} = 0 \quad \text{in } Q_\infty,\\
&\mathbf{z} = \mathbf{u}_s^\infty \text{ on } \Sigma_\infty, \quad \mathbf{z}(0) = \mathbf{w} + \mathbf{y}_0 \text{ in } \Omega.
\end{aligned}$$

Thus $\mathbf{y}_0$ is a perturbation of the stationary solution $\mathbf{w}$.

In [19], we have already studied this stabilization problem in two dimensions. Here, we want to study the extension of the results obtained in [19] to the three-dimensional case. Consider the linearized equation associated with (1.1):

$$\begin{aligned}
&\frac{\partial\mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} + (\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{w} + \nabla p = 0 \quad \text{in } Q_\infty,\\
&\operatorname{div}\mathbf{y} = 0 \text{ in } Q_\infty, \quad \mathbf{y} = M\mathbf{u} \text{ on } \Sigma_\infty, \quad \mathbf{y}(0) = \mathbf{y}_0 \text{ in } \Omega.
\end{aligned} \tag{1.2}$$

Following [18,19], this equation may be rewritten in the form

$$\begin{aligned}
&P\mathbf{y}' = AP\mathbf{y} + BM\mathbf{u} = AP\mathbf{y} + (\lambda_0 I - A)PD_A M\mathbf{u} \quad \text{in } (0,\infty), \quad P\mathbf{y}(0) = \mathbf{y}_0,\\
&(I-P)\mathbf{y} = (I-P)D_A M\mathbf{u} \quad \text{in } (0,\infty),
\end{aligned} \tag{1.3}$$

where P is the so-called Helmholtz or Leray projection operator, the operator A, with domain $D(A) = \mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega)\cap\mathbf{V}_n^0(\Omega)$, is the unbounded operator in $\mathbf{V}_n^0(\Omega)$ defined by $A\mathbf{y} = \nu P\Delta\mathbf{y} - P((\mathbf{w}\cdot\nabla)\mathbf{y}) - P((\mathbf{y}\cdot\nabla)\mathbf{w})$,

λ_0 belongs to the resolvent set of A, and D_A is the Dirichlet operator associated with $\lambda_0 I - A$ (see Section 2). In [19], the feedback control law is determined by solving the problem

$$\inf\big\{J(\mathbf{y},\mathbf{u}) \;\big|\; (\mathbf{y},\mathbf{u}) \text{ satisfies (1.2)}, \ \mathbf{u}\in L^2\big(0,\infty;\mathbf{V}^0(\Gamma)\big)\big\}, \tag{$\mathcal{R}$}$$

where

$$J(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty\int_\Omega |P\mathbf{y}|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^\infty\big(\big|\gamma_\tau\mathbf{u}(t)\big|^2_{\mathbf{V}^0(\Gamma)} + \big|R_A^{1/2}\gamma_n\mathbf{u}(t)\big|^2_{\mathbf{V}^0(\Gamma)}\big)\,\mathrm{d}t,$$

$\gamma_\tau\mathbf{u} = \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}$, $\gamma_n\mathbf{u} = (\mathbf{u}\cdot\mathbf{n})\mathbf{n}$, $R_A = MD_A^*(I-P)D_AM + I$, and

$$\mathbf{V}^0(\Gamma) = \bigg\{\mathbf{u}\in\mathbf{L}^2(\Gamma)\;\bigg|\;\int_\Gamma \mathbf{u}\cdot\mathbf{n} = 0\bigg\}.$$

Let $\Pi\in\mathcal{L}(\mathbf{V}_n^0(\Omega))$ be the solution to the Algebraic Riccati Equation associated with $(\mathcal{R})$. The closed loop system

$$\begin{aligned}
&P\mathbf{y}' = AP\mathbf{y} - BMR_A^{-1}MB^*\Pi P\mathbf{y}, \quad P\mathbf{y}(0) = \mathbf{y}_0,\\
&(I-P)\mathbf{y} = -(I-P)D_AR_A^{-1}MB^*\Pi P\mathbf{y},
\end{aligned}$$

admits a unique solution $\mathbf{y}_{\mathbf{y}_0}$ and $(\mathbf{y}_{\mathbf{y}_0}, -R_A^{-1}MB^*\Pi P\mathbf{y}_{\mathbf{y}_0})$ is the solution to problem $(\mathcal{R})$ (see [19]). If $A_\Pi = A - BMR_A^{-1}MB^*\Pi$ is the generator of the corresponding closed loop system, we have shown that the corresponding linear feedback law locally stabilizes Eq. (1.1). More precisely, the solution to the nonlinear system,

$$\begin{aligned}
&P\mathbf{y}' = A_\Pi P\mathbf{y} - P\big((\mathbf{y}\cdot\nabla)\mathbf{y}\big) \quad \text{in } (0,\infty), \quad \mathbf{y}(0) = \mathbf{y}_0,\\
&(I-P)\mathbf{y} = -(I-P)D_AMR_A^{-1}MB^*\Pi P\mathbf{y} \quad \text{in } (0,\infty),
\end{aligned} \tag{1.4}$$

obeys

$$\big|\mathbf{y}(t)\big|_{\mathbf{V}^{1/2-\varepsilon}(\Omega)} \leqslant C\mu,$$

for all initial condition satisfying $|\mathbf{y}_0|_{\mathbf{V}_n^{1/2-\varepsilon}(\Omega)} \leqslant C\mu$ for $\mu > 0$ small enough (here $\mathbf{V}_n^{1/2-\varepsilon}(\Omega) = \mathbf{V}_n^0(\Omega)\cap\mathbf{H}^{1/2-\varepsilon}(\Omega)$, see Section 2). An exponential decay of the form $C\mathrm{e}^{-\omega t}\mu$ may also be obtained by replacing Π by the operator Π_ω, where Π_ω is the solution to the Riccati equation in which the operator A is replaced by $A+\omega I$.

The analysis in [19] is based on the following properties:

(i) If $\mathbf{y}_0\in\mathbf{V}_n^0(\Omega)\cap\mathbf{H}^{1/2-\varepsilon}(\Omega)$, with $0<\varepsilon\leqslant 1/2$, then the optimal solution $(\bar{\mathbf{y}},\bar{\mathbf{u}})$ of $(\mathcal{R})$ belongs to $\mathbf{V}^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_\infty)\times\mathbf{V}^{3/2,3/4}(\Sigma_\infty)$.

(ii) Setting $F(\mathbf{y}) = -P((\mathbf{y}\cdot\nabla)\mathbf{y})$, the nonlinear mapping F is locally Lipschitz from $\mathbf{V}^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_\infty)$ into $L^1(0,\infty;\mathbf{V}_n^0(\Omega))\cap L^2(0,\infty;(\mathbf{V}^{2\varepsilon}(\Omega))')$ for all $0<\varepsilon<1/4$.

(iii) For $\lambda_0>0$ big enough, and all $0<\varepsilon<1/4$, the mapping $\mathbf{f}\mapsto\int_0^t \mathrm{e}^{(t-\tau)(A-\lambda_0)}\mathbf{f}(\tau)\,\mathrm{d}\tau$ is continuous from $L^1(0,\infty;\mathbf{V}_n^0(\Omega))\cap L^2(0,\infty;(\mathbf{V}^{2\varepsilon}(\Omega))')$ into $\mathbf{V}^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_\infty)$. (For the precise definition of the different spaces we refer to Section 2.)

With these properties and a fixed point method, we have been able to show that the linear feedback law also stabilizes, at least locally, the nonlinear system (1.1). In three dimension, property (ii) is no longer true. We can only prove that the nonlinear mapping F is locally Lipschitz from $\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)$ into $L^1(0,\infty;\mathbf{V}_n^0(\Omega))\cap L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')$ for all $0<\varepsilon\leqslant 1/2$. Therefore, to deal with the 3D case, we first have to look for a control problem for which the optimal state $P\mathbf{y}$ belongs to $\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)$ for some $0<\varepsilon\leqslant 1/2$. The solution $P\mathbf{y}$ to Eq. (1.3) belongs to this space only if $\mathbf{y}_0\in\mathbf{V}^{1/2+\varepsilon}(\Omega)$, $\mathbf{u}\in\mathbf{V}^{1+\varepsilon,1/2+\varepsilon/2}(\Sigma_\infty)$, and if $\mathbf{y}_0$ and $\mathbf{u}$ satisfy the compatibility condition $\mathbf{y}_0|_\Gamma = (M\mathbf{u})|_{t=0}$.

Thus to stabilize the three-dimensional Navier–Stokes system, the feedback control law and the initial condition have to satisfy some compatibility condition. Barbu, Lasiecka and Triggiani [5] have shown the existence of a feedback law, satisfying such a compatibility condition, by solving an optimal control problem with a cost functional involving the norm of $\mathbf{y}$ in the space $\mathbf{H}^{3/2+\varepsilon}(\Omega)$. Such an approach provides a Lyapunov functional for the associated closed loop dynamical system. But, as explained in [6], or in [19], and at the beginning of this introduction, the drawback of this method is that the Riccati equation, needed to calculate the feedback operator, is defined only in $D((A_\Pi)^2)$, where A_Π is the infinitesimal generator of the associated closed loop system. But $D((A_\Pi)^2)$ is not known because Π is not known. Therefore, in that case, the Riccati equation is not well defined, contrarily to what happens in the case of a distributed control (see e.g. [4]). (For another approach to construct Lyapunov functionals for semilinear parabolic equations in the one-dimensional case, we refer to [10].) Let us mention that the numerical approximation of the algebraic Riccati equation that we consider here (see below Eq. (1.6)) is studied in [3].

An alternative solution is proposed by Badra [1,2], where the compatibility condition between the initial condition and the control feedback law is guaranteed by solving an extended system. In [1,2], the boundary control is considered as a new state variable satisfying some equation on the boundary of the domain, and a source term in this equation is chosen as a new control variable. (This kind of controller is referred as a dynamic controller, see e.g. [11].) In that case, the Riccati equation is defined in a classical way, and the feedback law depends on the extended state.

Here, we are going to see that the compatibility condition between the initial condition and the control feedback law can be achieved by replacing the boundary and initial conditions

$$\mathbf{y}(t)|_\Gamma = M\mathbf{u}(t) \quad \text{and} \quad \mathbf{y}(0) = \mathbf{y}_0 \in \mathbf{V}_n^{1/2-\varepsilon}(\Omega), \quad 0 < \varepsilon < 1/4,$$

which are sufficient to deal with the two-dimensional case, by the following ones

$$\mathbf{y}(t)|_\Gamma = \theta(t)M\mathbf{u}(t) \quad \text{and} \quad \mathbf{y}(0) = \mathbf{y}_0 \in \mathbf{V}_0^{1/2+\varepsilon}(\Omega), \quad 0 < \varepsilon \leqslant 1/2,$$

where the nonnegative weight function θ is a regular function of t, taking values in $[0, 1]$, such that $\theta(0) = 0$ and $\theta(t) = 1$ for $t \geqslant t_0$ for some $t_0 > 0$. The role of θ is to guarantee that the compatibility condition

$$\mathbf{y}_0|_\Gamma = \theta M\mathbf{u}|_{t=0}$$

is always satisfied. By this way we can define a control problem similar to $(\mathcal{R})$ whose solutions belong to $\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)$. The drawback of introducing θ is that the feedback operator now depends on the time variable on the interval $[0, t_0]$, and it does not inherit the regularizing properties guaranteed by θ for the optimal state. We are going to see that one can improve the regularizing properties of the feedback operator by replacing the term $\int_0^\infty \int_\Omega |P\mathbf{y}|^2\,\mathrm{d}x\,\mathrm{d}t$ in the cost functional I by $\int_0^\infty \int_\Omega |(-A_0)^{-1/2}P\mathbf{y}|^2\,\mathrm{d}x\,\mathrm{d}t$, where $A_0 = P\Delta$ is the Stokes operator. Thus we have to study the control problem:

$$\inf\bigl\{I(\mathbf{y},\mathbf{u}) \bigm| (\mathbf{y},\mathbf{u}) \text{ satisfies (1.5)},\ \mathbf{u} \in \mathbf{V}^{0,0}(\Sigma_\infty)\bigr\}, \tag{$\mathcal{Q}_{\mathbf{y}_0}$}$$

where

$$I(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty\int_\Omega \bigl|(-A_0)^{-1/2}\mathbf{y}\bigr|^2 + \frac{1}{2}\int_0^\infty\int_\Gamma |\mathbf{u}|^2,$$

and

$$\mathbf{y}' = A_\omega\mathbf{y} + \theta BM\mathbf{u} = (A+\omega)\mathbf{y} + \theta BM\mathbf{u} \quad \text{in } (0,\infty), \quad \mathbf{y}(s) = \mathbf{y}_0, \tag{1.5}$$

with $\omega \geqslant 0$ given fixed. (In $(\mathcal{Q}_{\mathbf{y}_0})$, the state variable $\mathbf{y}$ plays the role of the velocity field $P\mathbf{y}$ solution of Eq. (1.3).) We show that problem $(\mathcal{Q}_{\mathbf{y}_0})$ admits a unique solution $(\mathbf{y}_{\mathbf{y}_0}, \mathbf{u}_{\mathbf{y}_0})$, which obeys the feedback formula

$$\mathbf{u}_{\mathbf{y}_0}(t) = -\theta(t)MB^*\Pi_\omega(t)\mathbf{y}_{\mathbf{y}_0}(t),$$

where $\Pi_\omega \in C_s([0,\infty); \mathcal{L}(\mathbf{V}_n^0(\Omega)))$ is the unique mapping satisfying,

$$\begin{aligned}
&\Pi_\omega^*(t) = \Pi_\omega(t) \in \mathcal{L}\bigl(\mathbf{V}_n^0(\Omega)\bigr) \quad \text{and} \quad \Pi_\omega(t) \geqslant 0 \quad \text{for all } t \geqslant 0,\\
&\text{for all } \mathbf{y} \in \mathbf{V}_n^0(\Omega), \quad \Pi_\omega(t)\mathbf{y} \in \mathbf{V}^2(\Omega) \cap \mathbf{V}_0^1(\Omega) \quad \text{and} \quad \bigl|\Pi_\omega(t)\mathbf{y}\bigr|_{\mathbf{V}^2(\Omega)} \leqslant C|\mathbf{y}|_{\mathbf{V}_n^0(\Omega)},
\end{aligned}$$

for $t \geqslant t_0$, $\Pi_\omega(t) = \widehat{\Pi}_\omega$, where $\widehat{\Pi}_\omega$ is the solution to the algebraic equation

$$\widehat{\Pi}_\omega = \widehat{\Pi}_\omega^* \geqslant 0, \quad A_\omega^*\widehat{\Pi}_\omega + \widehat{\Pi}_\omega A_\omega - \widehat{\Pi}_\omega BM^2B^*\widehat{\Pi}_\omega + (-A_0)^{-1} = 0 \tag{1.6}$$

for $t \leqslant t_0$, Π_ω is the solution to the differential equation

$$\begin{aligned} &-\Pi_\omega'(t) = A_\omega^* \Pi_\omega + \Pi_\omega A_\omega - \theta^2(t)\Pi_\omega B M^2 B^* \Pi_\omega + (-A_0)^{-1}, \\ &\Pi_\omega(t_0) = \widehat{\Pi}_\omega. \end{aligned} \tag{1.7}$$

If $A_{\omega,\Pi_\omega}(t)$, for $t \geqslant 0$, is the infinitesimal generator of the closed loop evolution operator corresponding to this new control problem, we show in Section 7 that the solution to the nonlinear evolution equation,

$$\begin{aligned} &P\mathbf{y}' = A_{\omega,\Pi_\omega}(t)P\mathbf{y} - P\big((\mathbf{y}\cdot\nabla)\mathbf{y}\big) \quad \text{in } (0,\infty), \quad P\mathbf{y}(0) = \mathbf{y}_0, \\ &(I-P)\mathbf{y} = -(I-P)\theta^2(\cdot)D_A M^2 B^* \Pi_\omega(\cdot)P\mathbf{y} \quad \text{in } (0,\infty), \end{aligned} \tag{1.8}$$

obeys

$$\big|\mathbf{y}(t)\big|_{\mathbf{V}^{1/2+\varepsilon}(\Omega)} \leqslant C_1(\mathbf{w},\varepsilon,\omega)\mathrm{e}^{-\omega t}\mu,$$

if $|\mathbf{y}_0|_{\mathbf{V}_0^{1/2+\varepsilon}(\Omega)} \leqslant C_0(\mathbf{w},\varepsilon)\mu$ and if $\mu > 0$ is small enough.

For notational simplicity, we perform the analysis of problem $(\mathcal{Q}_{\mathbf{y}_0})$ in the case where $\omega = 0$, and we denote by Π and $\widehat{\Pi}$ the solutions of (1.7) and (1.6) corresponding to $\omega = 0$. The extension to the case where $\omega > 0$ is treated at the end of Section 7.

To prove that the solution to the closed loop system (1.8) belongs to $\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)$, we have to show that, for all $s \in [0, 2t_0]$, the mapping $\Pi'(s)$ is bounded from $\mathbf{V}_n^0(\Omega)$ into $\mathcal{V}^2(\Omega) = \mathbf{V}^2(\Omega) \cap \mathbf{V}_0^1(\Omega)$ uniformly with respect to $s \in [0, 2t_0]$ (Theorem 5.6). We establish such a result by showing that

$\Pi(s)$ is bounded from $\mathcal{V}^{-2}(\Omega)$ to $\mathcal{V}^2(\Omega)$ uniformly w.r. to $s \in [0,\infty)$ (Corollary 5.3),

$\Pi(s)$ is bounded from $\mathbf{V}_n^{1/2-\varepsilon}(\Omega)$ to $\mathbf{V}^{9/2-\varepsilon}(\Omega)$ uniformly w.r. to $s \in [0,\infty)$ (Corollary 5.2).

On the interval $[t_0,\infty)$, Π is constant and equal to the solution $\widehat{\Pi}$ of Eq. (1.6). The properties of $\widehat{\Pi}$ are obtained by studying a problem, similar to $(\mathcal{Q}_{\mathbf{y}_0})$, in which we replace θ by the constant function equal to 1. This analysis is performed in Section 4. Problem $(\mathcal{Q}_{\mathbf{y}_0})$ is studied in Section 5. To deal with the nonlinear system, we analyze the regularity of solutions of the nonhomogeneous closed loop linear system:

$$\mathbf{y}' = A_\Pi(t)\mathbf{y} + \mathbf{f} \quad \text{in } (0,\infty), \quad \mathbf{y}(0) = \mathbf{y}_0. \tag{1.9}$$

The nonhomogeneous term $\mathbf{f}$ plays the role of the nonlinear term of the Navier–Stokes system. More precisely, we want to show that if $\mathbf{y}_0$ belongs to $\mathbf{V}_0^{1/2+\varepsilon}(\Omega)$ and if $\mathbf{f}$ belongs to $L^1(0,\infty;\mathbf{V}_n^0(\Omega)) \cap L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}(\Omega))')$, then the solution $\mathbf{y}$ to Eq. (1.9) belongs to $\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)$ (Lemma 6.3). This result is obtained by first studying a control problem with the nonhomogeneous term $\mathbf{f}$ in the state equation (see Section 6, Theorem 6.2). Next, we study Eq. (1.9) in Lemma 6.3. The local stabilization result of the Navier–Stokes equations is stated and proved in Section 7 (Theorem 7.1). We have collected some regularity results in Appendix A. The results of this paper have been announced in [21].

2. Functional framework and preliminary results

2.1. Notation and assumptions

Let us introduce the following spaces: $H^\sigma(\Omega;\mathbb{R}^N) = \mathbf{H}^\sigma(\Omega)$, $L^2(\Omega;\mathbb{R}^N) = \mathbf{L}^2(\Omega)$, the same notation conventions will be used for trace spaces and for the spaces $H_0^\sigma(\Omega;\mathbb{R}^N)$. Throughout what follows N is equal to 3. We also introduce different spaces of free divergence functions and some corresponding trace spaces:

$$\begin{aligned} &\mathbf{V}^\sigma(\Omega) = \big\{\mathbf{y} \in \mathbf{H}^\sigma(\Omega) \bigm| \operatorname{div}\mathbf{y} = 0 \text{ in } \Omega,\ \langle \mathbf{y}\cdot\mathbf{n}, 1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0\big\} \quad \text{for } \sigma \geqslant 0, \\ &\mathbf{V}_n^\sigma(\Omega) = \big\{\mathbf{y} \in \mathbf{H}^\sigma(\Omega) \bigm| \operatorname{div}\mathbf{y} = 0 \text{ in } \Omega,\ \mathbf{y}\cdot\mathbf{n} = 0 \text{ on } \Gamma\big\} \quad \text{for } \sigma \geqslant 0, \\ &\mathbf{V}_0^\sigma(\Omega) = \big\{\mathbf{y} \in \mathbf{H}^\sigma(\Omega) \bigm| \operatorname{div}\mathbf{y} = 0 \text{ in } \Omega,\ \mathbf{y} = 0 \text{ on } \Gamma\big\} \quad \text{for } \sigma > 1/2, \\ &\mathbf{V}^\sigma(\Gamma) = \big\{\mathbf{y} \in \mathbf{H}^\sigma(\Gamma) \bigm| \langle \mathbf{y}\cdot\mathbf{n}, 1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0\big\} \quad \text{for } \sigma \geqslant -1/2. \end{aligned}$$

In the above setting $\mathbf{n}$ denotes the unit normal to Γ outward Ω. The spaces $\mathbf{V}^\sigma(\Omega)$ and $\mathbf{V}^\sigma(\Gamma)$ are respectively equipped with the usual norms of $\mathbf{H}^\sigma(\Omega)$ and $\mathbf{H}^\sigma(\Gamma)$, these norms will be denoted by $|\cdot|_{\mathbf{V}^\sigma(\Omega)}$ and $|\cdot|_{\mathbf{V}^\sigma(\Gamma)}$.

We shall use the following notation $Q_T = \Omega \times (0,T)$, $\Sigma_T = \Gamma \times (0,T)$, $Q_{\bar t,T} = \Omega \times (\bar t, T)$ and $\Sigma_{\bar t,T} = \Gamma \times (\bar t, T)$ for $\bar t > 0$, and $0 < T \leqslant \infty$. For spaces of time dependent functions we set

$$\mathbf{V}^{\sigma,\tilde\sigma}(Q_T) = H^{\tilde\sigma}\big(0,T;\mathbf{V}^0(\Omega)\big) \cap L^2\big(0,T;\mathbf{V}^\sigma(\Omega)\big),$$

and

$$\mathbf{V}^{\sigma,\tilde\sigma}(\Sigma_T) = H^{\tilde\sigma}\big(0,T;\mathbf{V}^0(\Gamma)\big) \cap L^2\big(0,T;\mathbf{V}^\sigma(\Gamma)\big).$$

Let P be the orthogonal projection in $\mathbf{L}^2(\Omega)$ onto $\mathbf{V}^0_n(\Omega)$. The Stokes operator is defined by $D(A_0) = \mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega)$ and $A_0 = P\Delta$. We also introduce the spaces corresponding to the domains of fractional powers of $(-A_0)$, and we set:

$$\mathcal{V}^\sigma(\Omega) = D\big((-A_0)^{\sigma/2}\big) \text{ if } \sigma \geqslant 0, \quad \text{and} \quad \mathcal{V}^\sigma(\Omega) = \big(D\big((-A_0)^{-\sigma/2}\big)\big)' \text{ if } \sigma < 0.$$

Thus we have $\mathcal{V}^2(\Omega) = \mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega)$, $\mathcal{V}^1(\Omega) = \mathbf{V}^1_0(\Omega)$, $\mathcal{V}^{-2}(\Omega) = (\mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega))'$, and $\mathcal{V}^{-1}(\Omega) = (\mathbf{V}^1_0(\Omega))'$. For $\sigma \geqslant 0$, $\mathcal{V}^\sigma(\Omega)$ will be equipped with the norm of $\mathbf{H}^\sigma(\Omega)$, and for $\sigma < 0$, $\mathcal{V}^\sigma(\Omega)$ will be equipped with the dual norm of $\mathcal{V}^{-\sigma}(\Omega)$.

We assume that Ω is of class C^4 and $\mathbf{w} \in \mathbf{V}^3(\Omega)$.

In order to find a control $\mathbf{u}$, supported in an open subset Γ_c of Γ, we introduce a weight function $m \in C^4(\Gamma)$ with values in [0, 1], with support in Γ_c, equal to 1 in Γ_0, where Γ_0 is an open subset in Γ_c. Associated with this function m we introduce the operator $M \in \mathcal{L}(\mathbf{V}^0(\Gamma))$ defined by

$$M\mathbf{u}(x) = m(x)\mathbf{u}(x) - \frac{m}{\int_\Gamma m}\bigg(\int_\Gamma m\mathbf{u}\cdot\mathbf{n}\bigg)\mathbf{n}(x).$$

By this way, we can replace the condition $\operatorname{supp}(\mathbf{u}) \subset \Gamma_c$ by considering a boundary condition of the form

$$\mathbf{z} - \mathbf{w} = M\mathbf{u} \quad \text{on } \Sigma_\infty.$$

The main interest of this operator M is that if $\mathbf{u} \in L^2(0,\infty; H^\sigma(\Gamma_c;\mathbb{R}^N)) \cap H^{\sigma/2}(0,\infty; L^2(\Gamma_c;\mathbb{R}^N))$ for some $0 < \sigma \leqslant 2$, and if $\tilde{\mathbf{u}}$ denotes the extension of $\mathbf{u}$ by zero to $\Sigma_\infty \setminus (\Gamma_c \times (0,\infty))$, then $M\tilde{\mathbf{u}}$ belongs to $L^2(0,\infty; H^\sigma(\Gamma;\mathbb{R}^N)) \cap H^{\sigma/2}(0,\infty; L^2(\Gamma;\mathbb{R}^N))$, which is not true for $\tilde{\mathbf{u}}$.

For all $\psi \in H^{1/2+\varepsilon'}(\Omega)$, with $\varepsilon' > 0$, we denote by $c(\psi)$ the constant defined by:

$$c(\psi) = \frac{1}{|\Gamma|}\int_\Gamma \psi, \tag{2.1}$$

where $|\Gamma|$ is the $(N-1)$-dimensional Lebesgue measure of Γ.

2.2. Properties of some operators

Let us denote by $(A, D(A))$ and $(A^*, D(A^*))$, the unbounded operators in $\mathbf{V}^0_n(\Omega)$ defined by:

$$\begin{aligned} &D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{V}^1_0(\Omega), \quad A\mathbf{y} = \nu P\Delta\mathbf{y} - P\big((\mathbf{w}\cdot\nabla)\mathbf{y}\big) - P\big((\mathbf{y}\cdot\nabla)\mathbf{w}\big), \\ &D(A^*) = \mathbf{H}^2(\Omega) \cap \mathbf{V}^1_0(\Omega), \quad A^*\mathbf{y} = \nu P\Delta\mathbf{y} + P\big((\mathbf{w}\cdot\nabla)\mathbf{y}\big) - P\big((\nabla\mathbf{w})^T\mathbf{y}\big). \end{aligned}$$

Throughout the following we denote by $\lambda_0 > 0$ an element in the resolvent set of A satisfying

$$\begin{aligned} &\big(\big(\lambda_0(-A_0)^{-\alpha} - A\big)\mathbf{y},\mathbf{y}\big)_{\mathbf{V}^0_n(\Omega)} \geqslant \omega_0|\mathbf{y}|^2_{\mathbf{V}^1_0(\Omega)} \quad \text{for all } \mathbf{y} \in D(A), \quad \text{and} \\ &\big(\big(\lambda_0(-A_0)^{-\alpha} - A^*\big)\mathbf{y},\mathbf{y}\big)_{\mathbf{V}^0_n(\Omega)} \geqslant \omega_0|\mathbf{y}|^2_{\mathbf{V}^1_0(\Omega)} \quad \text{for all } \mathbf{y} \in D(A^*), \end{aligned} \tag{2.2}$$

for some $0 < \omega_0 < \nu$, and all $0 \leqslant \alpha \leqslant 1/2$ (see [20, Lemma 24]). Let us recall two results from [18] and [20].

Theorem 2.1. *[20, Lemma 25] For all $0 \leqslant \alpha \leqslant 1/2$, the unbounded operator $(A - \lambda_0(-A_0)^{-\alpha})$ (respectively $(A^* - \lambda_0(-A_0)^{-\alpha})$) with domain $D(A - \lambda_0(-A_0)^{-\alpha}) = D(A)$ (respectively $D(A^* - \lambda_0(-A_0)^{-\alpha}) = D(A^*)$) is the infinitesimal generator of an exponentially stable analytic semigroup on $\mathbf{V}_n^0(\Omega)$. Moreover, for all $0 \leqslant \beta \leqslant 1$, we have*

$$D\big(\big(\lambda_0(-A_0)^{-\alpha} - A\big)^\beta\big) = D\big(\big(\lambda_0(-A_0)^{-\alpha} - A^*\big)^\beta\big) = D\big(\big(\lambda_0(-A_0)^{-\alpha} - A_0\big)^\beta\big) = D\big((-A_0)^\beta\big).$$

Let us introduce D_A and D_p, two Dirichlet operators associated with A, defined as follows. For $\mathbf{u} \in \mathbf{V}^0(\Gamma)$, set $D_A\mathbf{u} = \mathbf{y}$ and $D_p\mathbf{u} = q$ where $(\mathbf{y}, q)$ is the unique solution in $\mathbf{V}^{1/2}(\Omega) \times (H^{1/2}(\Omega)/\mathbb{R})'$ to the equation

$$\begin{aligned} &\lambda_0\mathbf{y} - \nu\Delta\mathbf{y} + (\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{w} + \nabla q = 0 \quad \text{in } \Omega, \\ &\operatorname{div}\mathbf{y} = 0 \text{ in } \Omega, \quad \mathbf{y} = \mathbf{u} \text{ on } \Gamma. \end{aligned}$$

Lemma 2.1. *[18, Corollary 7.1 and Lemma 7.4] The operator D_A is a bounded operator from $\mathbf{V}^0(\Gamma)$ into $\mathbf{V}^0(\Omega)$, moreover it satisfies*

$$|D_A\mathbf{u}|_{\mathbf{V}^{\sigma+1/2}(\Omega)} \leqslant C(\sigma)|\mathbf{u}|_{\mathbf{V}^\sigma(\Gamma)} \quad \text{for all } 0 \leqslant \sigma \leqslant 7/2.$$

The operator $D_A^* \in \mathcal{L}(\mathbf{V}^0(\Omega), \mathbf{V}^0(\Gamma))$, the adjoint operator of $D_A \in \mathcal{L}(\mathbf{V}^0(\Gamma), \mathbf{V}^0(\Omega))$, is defined by:

$$D_A^*\mathbf{g} = -\nu\frac{\partial\mathbf{z}}{\partial\mathbf{n}} + \pi\mathbf{n} - c(\pi)\mathbf{n}, \tag{2.3}$$

where $(\mathbf{z}, \pi)$ is the solution of

$$\lambda_0\mathbf{z} - \nu\Delta\mathbf{z} - (\mathbf{w}\cdot\nabla)\mathbf{z} + (\nabla\mathbf{w})^T\mathbf{z} + \nabla\pi = \mathbf{g} \quad \textit{and} \quad \operatorname{div}\mathbf{z} = 0 \textit{ in } \Omega, \quad \mathbf{z} = 0 \textit{ on } \Gamma, \tag{2.4}$$

and $c(\pi)$ is the constant corresponding to π, defined in (2.1).

Let us define the operators $\gamma_\tau \in \mathcal{L}(\mathbf{V}^0(\Gamma))$ and $\gamma_n \in \mathcal{L}(\mathbf{V}^0(\Gamma))$ by

$$\gamma_\tau\mathbf{u} = \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \quad \text{and} \quad \gamma_n\mathbf{u} = (\mathbf{u}\cdot\mathbf{n})\mathbf{n} = \mathbf{u} - \gamma_\tau\mathbf{u} \quad \text{for all } \mathbf{u} \in \mathbf{V}^0(\Gamma).$$

Let us also denote by P_Γ the projector from $\mathbf{L}^2(\Gamma)$ onto $\mathbf{V}^0(\Gamma)$ defined by $P_\Gamma\mathbf{u} = \mathbf{u} - \frac{m}{\int_\Gamma m}(\int_\Gamma \mathbf{u}\cdot\mathbf{n})\mathbf{n}$. Observe that $M = P_\Gamma m$, where m denotes the multiplication operator by the function m.

Lemma 2.2. *[19, Lemma 2.4] The operator M obeys the following properties:*

$$M = M^*, \quad M\gamma_\tau = \gamma_\tau M = m\gamma_\tau, \quad \textit{and} \quad M\gamma_n = \gamma_n M.$$

The operators γ_τ and γ_n satisfy:

$$\gamma_\tau = \gamma_\tau^*, \quad \gamma_n = \gamma_n^* \quad \textit{and} \quad (I - P)D_A = (I - P)D_A\gamma_n.$$

We introduce the operators

$$B_n = (\lambda_0 I - A)PD_A\gamma_n, \quad B_\tau = (\lambda_0 I - A)D_A\gamma_\tau, \quad B = B_n + B_\tau.$$

Proposition 2.1. *[19, Lemma 2.3] For all $\Phi \in D(A^*)$, $B^*\Phi$ belongs to $\mathbf{V}^{1/2}(\Gamma)$, we have*

$$B^*\Phi = D_A^*(\lambda_0 I - A^*)\Phi, \quad B_\tau^*\Phi = \gamma_\tau D_A^*(\lambda_0 I - A^*)\Phi, \quad B_n^*\Phi = \gamma_n D_A^*(\lambda_0 I - A^*)\Phi,$$

and

$$B^*\Phi = -\nu\frac{\partial\Phi}{\partial\mathbf{n}} + \psi\mathbf{n} - c(\psi)\mathbf{n}, \quad B_\tau^*\Phi = -\nu\frac{\partial\Phi}{\partial\mathbf{n}}, \quad B_n^*\Phi = \psi\mathbf{n} - c(\psi)\mathbf{n},$$

with

$$\nabla\psi = (I - P)\big[\nu\Delta\Phi + (\mathbf{w}\cdot\nabla)\Phi - (\nabla\mathbf{w})^T\Phi\big],$$

and $c(\psi)$ is defined by (2.1). *In particular if $\Phi \in \mathbf{V}^\sigma(\Omega) \cap \mathbf{V}_0^1(\Omega)$ with $\sigma > 3/2$, the following estimate holds*

$$|B^*\Phi|_{\mathbf{V}^{\sigma-3/2}(\Gamma)} \leqslant C|\Phi|_{\mathbf{V}^\sigma(\Omega)\cap\mathbf{V}_0^1(\Omega)}.$$

3. Finite time horizon control problems

As already mentioned in the introduction, in [19] we have determined a feedback control law able to stabilize Eq. (1.3), by considering the family of control problems:

$$\inf\big\{J_T(s,\mathbf{y},\mathbf{u}) \;\big|\; (\mathbf{y},\mathbf{u}) \text{ satisfies (3.1)},\ \mathbf{u}\in L^2\big(s,T;\mathbf{V}^0(\Gamma)\big)\big\}, \qquad (\mathcal{R}^T_{s,\zeta})$$

where

$$J_T(s,\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_s^T\int_\Omega |P\mathbf{y}|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_s^T\big(\big|\gamma_\tau\mathbf{u}(t)\big|^2_{\mathbf{V}^0(\Gamma)} + \big|R_A^{1/2}\gamma_n\mathbf{u}(t)\big|^2_{\mathbf{V}^0(\Gamma)}\big)\,\mathrm{d}t,$$

and

$$\begin{aligned} &P\mathbf{y}' = AP\mathbf{y} + BM\mathbf{u} = AP\mathbf{y} + (\lambda_0 I - A)PD_AM\mathbf{u} \quad \text{in } (s,T), \quad P\mathbf{y}(s)=\zeta,\\ &(I-P)\mathbf{y} = (I-P)D_AM\mathbf{u} \quad \text{in } (s,T). \end{aligned} \tag{3.1}$$

Let us recall that $R_A = MD_A^*(I-P)D_AM + I$. Observe that, due to the definition of R_A, we have:

$$J_T(s,\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_s^T\int_\Omega |\mathbf{y}|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_s^T\big|\mathbf{u}(t)\big|^2_{\mathbf{V}^0(\Gamma)}\,\mathrm{d}t,$$

if $(\mathbf{y},\mathbf{u})$ is a solution of (3.1). Here we want to consider a new class of problems by replacing the term $\frac{1}{2}\int_s^T\int_\Omega |P\mathbf{y}|^2\,\mathrm{d}x\,\mathrm{d}t$ by $\frac{1}{2}\int_s^T\int_\Omega |(-A_0)^{-1/2}P\mathbf{y}|^2\,\mathrm{d}x\,\mathrm{d}t$. Thus the corresponding functional is

$$\mathcal{J}_T(s,\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_s^T\int_\Omega \big|(-A_0)^{-1/2}P\mathbf{y}\big|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_s^T\big(\big|\gamma_\tau\mathbf{u}(t)\big|^2_{\mathbf{V}^0(\Gamma)} + \big|R_A^{1/2}\gamma_n\mathbf{u}(t)\big|^2_{\mathbf{V}^0(\Gamma)}\big)\,\mathrm{d}t.$$

Actually, since R_A is an automorphism in $\{\mathbf{u}\in\mathbf{V}^0(\Gamma) \mid \gamma_\tau\mathbf{u}=0\}$, we can also consider the functional

$$\mathcal{I}_T(s,\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_s^T\int_\Omega \big|(-A_0)^{-1/2}P\mathbf{y}\big|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_s^T\big|\mathbf{u}(t)\big|^2_{\mathbf{V}^0(\Gamma)}\,\mathrm{d}t.$$

Replacing $P\mathbf{y}$ by $\mathbf{y}$ (for notational simplicity), we have to study the following family of finite time horizon control problems,

$$\inf\big\{I_T(s,\mathbf{y},\mathbf{u}) \;\big|\; (\mathbf{y},\mathbf{u}) \text{ satisfies (3.2)},\ \mathbf{u}\in\mathbf{V}^{0,0}(\Sigma_{s,T})\big\}, \qquad (\mathcal{P}^T_{s,\zeta})$$

where

$$\mathbf{y}' = A\mathbf{y} + BM\mathbf{u} \quad \text{in } (s,T), \quad \mathbf{y}(s)=\zeta, \tag{3.2}$$

and

$$I_T(s,\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_s^T\int_\Omega \big|(-A_0)^{-1/2}\mathbf{y}\big|^2 + \frac{1}{2}\int_s^T\int_\Gamma |\mathbf{u}|^2.$$

3.1. Problem $(\mathcal{P}^T_{s,\zeta})$ with initial conditions in $\mathbf{V}^0_n(\Omega)$

In this section we recall some results from [20].

Theorem 3.1. *For all $s\in[0,T]$ and all $\zeta\in\mathbf{V}^0_n(\Omega)$, problem $(\mathcal{P}^T_{s,\zeta})$ admits a unique solution $(\mathbf{y}^s_\zeta,\mathbf{u}^s_\zeta)$. The optimal control $\mathbf{u}^s_\zeta$ is characterized by*

$$\mathbf{u}^s_\zeta = -MB^*\Phi^s_\zeta \quad \textit{in } (s,T), \tag{3.3}$$

where Φ_ζ^s is solution to the equation

$$-\Phi' = A^*\Phi + (-A_0)^{-1}\mathbf{y}_\zeta^s \quad \text{in } (s,T), \quad \Phi(T) = 0. \tag{3.4}$$

Conversely the system

$$\begin{aligned} &\mathbf{y}' = A\mathbf{y} - BM^2B^*\Phi \quad \text{in } (s,T), \quad \mathbf{y}(s) = \zeta, \\ &-\Phi' = A^*\Phi + (-A_0)^{-1}\mathbf{y} \quad \text{in } (s,T), \quad \Phi(T) = 0, \end{aligned} \tag{3.5}$$

admits a unique solution $(\mathbf{y}_\zeta^s, \Phi_\zeta^s)$ in $L^2(s,T;\mathbf{V}_n^0(\Omega)) \times (\mathbf{V}^{2,1}(Q_{s,T}) \cap L^2(s,T;\mathbf{V}_0^1(\Omega))$, and $(\mathbf{y}_\zeta^s, -MB^*\Phi_\zeta^s)$ is the optimal solution to $(\mathcal{P}_{s,\zeta}^T)$.

Proof. See [20, Theorem 7]. (See also the proof of Theorem 3.4 where a similar result is proved in the case where $\zeta \in \mathcal{V}^{-2}(\Omega)$.) □

In the following theorem we improve the regularity result of the optimal solution (see [20, Theorem 8]).

Theorem 3.2. *The solution $(\mathbf{y}_\zeta^s, \Phi_\zeta^s)$ to system* (3.5) *belongs to $\mathbf{V}^{1,1/2}(Q_{s,T}) \times L^2(s,T;\mathbf{V}^{7/2-\varepsilon}(\Omega)) \cap H^{3/2}(s,T;\mathbf{V}_n^{1/2-\varepsilon}(\Omega))$ for all $\varepsilon > 0$. In particular Φ_ζ^s belongs to $C([s,T];\mathcal{V}^2(\Omega))$.*

Corollary 3.1. *For all $s \in [0,T]$ and all $\zeta \in \mathbf{V}_n^0(\Omega)$, the unique solution $(\mathbf{y}_\zeta^s, \mathbf{u}_\zeta^s)$ to problem $(\mathcal{P}_{s,\zeta}^T)$ and the corresponding solution $(\mathbf{y}_\zeta^s, \Phi_\zeta^s)$ to system* (3.5) *obey*

$$I_T(s, \mathbf{y}_\zeta^s, \mathbf{u}_\zeta^s) = \frac{1}{2}\int_\Omega \Phi_\zeta^s(s)\cdot\zeta.$$

Proof. See [20, Corollary 9]. □

Let $\Pi(s)$ be the operator defined by

$$\Pi(s) : \zeta \mapsto \Phi_\zeta^s(s), \tag{3.6}$$

where $(\mathbf{y}_\zeta^s, \Phi_\zeta^s)$ is the unique solution to system (3.5). From Theorem 3.2 it follows that $\Pi(s) \in \mathcal{L}(\mathbf{V}_n^0(\Omega), \mathcal{V}^2(\Omega))$. We can prove that the family of operators $(\Pi(s))_{s\in[0,T]}$ defined by (3.6) belongs to $C_s([0,T];\mathcal{L}(\mathbf{V}_n^0(\Omega)))$ (the space of functions Π from $[0,T]$ into $\mathcal{L}(\mathbf{V}_n^0(\Omega))$ such that, for all $\mathbf{y} \in \mathbf{V}_n^0(\Omega)$, $\Pi(\cdot)\mathbf{y}$ is continuous from $[0,T]$ into $\mathbf{V}_n^0(\Omega)$). Next, using the optimality system (3.5) we can show that Π is the unique weak solution in $C_s([0,T];\mathcal{L}(\mathbf{V}_n^0(\Omega))$ to the Riccati equation

$$\begin{aligned} &\Pi^*(t) = \Pi(t) \quad \text{and} \quad \Pi(t) \geqslant 0, \\ &\text{for all } \mathbf{y} \in \mathbf{V}_n^0(\Omega),\ t \in [0,T], \quad \Pi(t)\mathbf{y} \in \mathcal{V}^2(\Omega) \quad \text{and} \quad \left|\Pi(t)\mathbf{y}\right|_{\mathcal{V}^2(\Omega)} \leqslant C|\mathbf{y}|_{\mathbf{V}_n^0(\Omega)}, \\ &-\Pi'(t) = A^*\Pi(t) + \Pi(t)A - \Pi(t)BM^2B^*\Pi(t) + (-A_0)^{-1}, \\ &\Pi(T) = 0. \end{aligned} \tag{3.7}$$

From the definition of Π, from Theorem 3.1 and Corollary 3.1 we deduce the following theorem.

Theorem 3.3. *[20, Theorem 10] The solution $(\mathbf{y}, \mathbf{u})$ to problem $(\mathcal{P}_{0,\mathbf{y}_0}^T)$ belongs to $C([0,T];\mathbf{V}_n^0(\Omega)) \times C([0,T];\mathbf{V}^0(\Gamma))$, it obeys the feedback formula*

$$\mathbf{u}(t) = -MB^*\Pi(t)\mathbf{y}(t),$$

and the optimal cost is given by

$$J(\mathbf{y}, \mathbf{u}) = \frac{1}{2}\big(\Pi(0)\mathbf{y}_0, \mathbf{y}_0\big)_{\mathbf{V}_n^0(\Omega)}.$$

If we set $\overline{\Pi}(t)=\Pi(T-t)$, then $\overline{\Pi}$ is the unique solution in $C_s([0,T];\mathcal{L}(\mathbf{V}_n^0(\Omega)))$ to the Riccati equation

$$\begin{aligned}
&\overline{\Pi}^*(t)=\overline{\Pi}(t) \quad \text{and} \quad \overline{\Pi}(t)\geqslant 0,\\
&\text{for all } \mathbf{y}\in \mathbf{V}_n^0(\Omega),\ t\in[0,T], \quad \overline{\Pi}(t)\mathbf{y}\in\mathcal{V}^2(\Omega) \quad \text{and} \quad \big|\overline{\Pi}(t)\mathbf{y}\big|_{\mathcal{V}^2(\Omega)}\leqslant C|\mathbf{y}|_{\mathbf{V}_n^0(\Omega)},\\
&\overline{\Pi}'(t)=A^*\overline{\Pi}(t)+\overline{\Pi}(t)A-\overline{\Pi}(t)BM^2B^*\overline{\Pi}(t)+(-A_0)^{-1},\\
&\overline{\Pi}(0)=0.
\end{aligned}\tag{3.8}$$

From the definition of $\overline{\Pi}$ it follows that $\Pi(0)=\overline{\Pi}(T)$.

3.2. Problem $(\mathcal{P}_{s,\zeta}^T)$ with initial conditions in $\mathcal{V}^{-2}(\Omega)$

As already mentioned in the introduction, we have to study a time dependent feedback operator. The corresponding closed loop system:

$$\mathbf{y}'=A\mathbf{y}-\theta^2BM^2B^*\Pi\mathbf{y} \quad \text{in } (0,\infty), \quad \mathbf{y}(s)=\mathbf{y}_0,$$

is introduced in Section 5. If $\mathbf{y}_0\in\mathbf{V}_n^0(\Omega)$, the solution $\mathbf{y}_{\mathbf{y}_0}^s$ to this equation belongs to $C^1([s,\infty);\mathcal{V}^{-2}(\Omega))$ (see Remark 5.1). In the above equation, the operator Π depends on the time variable t. In order to study some Lipschitz properties of this operator (see Corollary 5.4), we have to show that, for all $t\in[s,t_0]$, $\Pi(t)\zeta$ is well defined when $\zeta\in\mathcal{V}^{-2}(\Omega)$. Actually we want to estimate $\Pi(t)\zeta$ when $\zeta=(\mathbf{y}_{\mathbf{y}_0}^s)'(s)\in\mathcal{V}^{-2}(\Omega)$. For that, we are going to study problem $(\mathcal{P}_{s,\zeta}^T)$ in the case where $\zeta\in\mathcal{V}^{-2}(\Omega)$.

Let us first study Eq. (3.2) when $\zeta\in\mathcal{V}^{-2}(\Omega)$ and $\mathbf{u}\in L^2(s,T;\mathbf{V}^0(\Gamma))$. Notice that the weak solution $\mathbf{y}_i$ to equation

$$\mathbf{y}_i'=A\mathbf{y}_i \quad \text{in } (s,T), \quad \mathbf{y}_i(s)=\zeta,\tag{3.9}$$

with $\zeta\in\mathcal{V}^{-2}(\Omega)$, is defined in [7, p. 167] via the extrapolation method. In the case when $\zeta\in\mathcal{V}^{-2}(\Omega)$, $\mathbf{y}_i$ is a weak solution to Eq. (3.9) if and only if $(\lambda_0I-A)^{-1}\mathbf{y}_i=\mathbf{z}$ is the solution of

$$\mathbf{z}'=A\mathbf{z} \quad \text{in } (s,T), \quad \mathbf{z}(s)=(\lambda_0I-A)^{-1}\zeta.$$

Since $\mathbf{z}$ obeys the estimate

$$\|\mathbf{z}\|_{L^2(s,T;\mathbf{V}_0^1(\Omega))}+\|\mathbf{z}\|_{H^1(s,T;\mathcal{V}^{-1}(\Omega))}\leqslant C\big|(\lambda_0I-A)^{-1}\zeta\big|_{\mathbf{V}_n^0(\Omega)},$$

we deduce that

$$\|\mathbf{y}_i\|_{L^2(s,T;\mathcal{V}^{-1}(\Omega))}+\|\mathbf{y}_i\|_{H^1(s,T;\mathcal{V}^{-3}(\Omega))}\leqslant C|\zeta|_{\mathcal{V}^{-2}(\Omega)}.$$

Therefore Eq. (3.2) admits a unique weak solution $\mathbf{y}$ in $L^2(s,T;\mathcal{V}^{-1}(\Omega))$, $\mathbf{y}=\mathbf{y}_i+\mathbf{y}_b$, where $\mathbf{y}_i(t)=\mathrm{e}^{(t-s)A}\zeta$, $\mathbf{y}_b(t)=\int_s^t\mathrm{e}^{(t-\tau)A}BM\mathbf{u}(\tau)\,\mathrm{d}\tau$, and

$$\begin{aligned}
&\|\mathbf{y}_i\|_{L^2(s,T;\mathcal{V}^{-1}(\Omega))}+\|\mathbf{y}_i\|_{H^1(s,T;\mathcal{V}^{-3}(\Omega))}\leqslant C|\zeta|_{\mathcal{V}^{-2}(\Omega)},\\
&\|\mathbf{y}_b\|_{\mathbf{V}^{1/2,1/4}(Q_{s,T})}\leqslant C\|\mathbf{u}\|_{L^2(s,T;\mathbf{V}^0(\Gamma))}
\end{aligned}\tag{3.10}$$

(see Lemmas A.1 and A.3, and Remark A.1).

Theorem 3.4. *For all $s\in[0,T]$, and all $\zeta\in\mathcal{V}^{-2}(\Omega)$, problem $(\mathcal{P}_{s,\zeta}^T)$ admits a unique solution $(\mathbf{y}_\zeta^s,\mathbf{u}_\zeta^s)$. The optimal control $\mathbf{u}_\zeta^s$ is characterized by*:

$$\mathbf{u}_\zeta^s=-MB^*\Phi_\zeta^s \quad \textit{in } (s,T),\tag{3.11}$$

where Φ_ζ^s is solution to the equation

$$-\Phi'=A^*\Phi+(-A_0)^{-1}\mathbf{y}_\zeta^s \quad \textit{in } (s,T), \quad \Phi(T)=0.\tag{3.12}$$

Conversely the system

$$\begin{aligned}&\mathbf{y}' = A\mathbf{y} - BM^2B^*\Phi \quad \text{in } (s,T), \quad \mathbf{y}(s) = \zeta,\\ &-\Phi' = A^*\Phi + (-A_0)^{-1}\mathbf{y} \quad \text{in } (s,T), \quad \Phi(T) = 0,\end{aligned} \tag{3.13}$$

admits a unique solution $(\mathbf{y}^s_\zeta, \Phi^s_\zeta)$ *in* $L^2(s,T;\mathcal{V}^{-1}(\Omega)) \times (\mathbf{V}^{2,1}(Q_{s,T}) \cap L^2(s,T;\mathbf{V}^1_0(\Omega)))$, *and* $(\mathbf{y}^s_\zeta, -MB^*\Phi^s_\zeta)$ *is the optimal solution to* $(\mathcal{P}^T_{s,\zeta})$.

The following estimate holds

$$\|\mathbf{y}^s_\zeta\|_{L^2(s,T;\mathcal{V}^{-1}(\Omega))} + \|\Phi^s_\zeta\|_{L^2(s,T;\mathbf{V}^3(\Omega)) \cap H^1(s,T;\mathbf{V}^1_0(\Omega))} \leqslant C|\zeta|_{\mathcal{V}^{-2}(\Omega)}, \tag{3.14}$$

for all $s \in [0,T]$, *and all* $\zeta \in \mathcal{V}^{-2}(\Omega)$. (*The constant* C *depends on* T, *but is independent of* $s \in [0,T]$.)

Proof. The proof is similar to that of [19, Theorem 3.1]. Since we deal with control problems with initial conditions in $\mathcal{V}^{-2}(\Omega)$, we rewrite the proof for the convenience of the reader.

Step 1. The existence of a unique solution $(\mathbf{y}^s_\zeta, \mathbf{u}^s_\zeta)$ to problem $(\mathcal{P}^T_{s,\zeta})$ is obvious. Let $\mathbf{u}$ be in $L^2(s,T;\mathbf{V}^0(\Gamma))$ and $\mathbf{v} \in L^2(s,T;\mathbf{V}^0(\Gamma))$. Assume that ζ is given fixed in $\mathcal{V}^{-2}(\Omega)$, denote by $\mathbf{y_u}$ the solution to Eq. (3.2) corresponding to $\mathbf{u}$, and set

$$I_T(s, \mathbf{y_u}, \mathbf{u}) = \mathbf{I}_T(\mathbf{u}).$$

We have

$$\mathbf{I}'_T(\mathbf{u})\mathbf{v} = \int_s^T\int_\Omega (-A_0)^{-1}\mathbf{y_u}\cdot\mathbf{z} + \int_s^T\int_\Gamma \mathbf{u}\cdot\mathbf{v},$$

where $\mathbf{z}$ is the solution to

$$\mathbf{z}' = A\mathbf{z} + BM\mathbf{v} \quad \text{in } (s,T), \quad \mathbf{z}(s) = 0.$$

Let Φ be the solution to the equation

$$-\Phi' = A^*\Phi + (-A_0)^{-1}\mathbf{y_u}, \quad \Phi(T) = 0.$$

Since $(-A_0)^{-1}\mathbf{y_u} \in L^2(s,T;\mathbf{V}^0_n(\Omega))$, due to Lemma A.8 with $2\alpha = 0$, Φ belongs to $\mathbf{V}^{2,1}(Q_{s,T})$. Thus $B^*\Phi$ belongs to $L^2(s,T;\mathbf{V}^0(\Gamma))$. The functions $\mathbf{z}$ and Φ obey the following identity:

$$\int_s^T\int_\Omega (-A_0)^{-1}\mathbf{y_u}\cdot\mathbf{z} = \int_s^T\int_\Gamma \mathbf{v}\cdot MB^*\Phi.$$

Thus

$$\mathbf{I}'_T(\mathbf{u})\mathbf{v} = \int_s^T\int_\Gamma \mathbf{v}\cdot MB^*\Phi + \int_s^T\int_\Gamma \mathbf{u}\cdot\mathbf{v}. \tag{3.15}$$

If $(\mathbf{y}^s_\zeta, \mathbf{u}^s_\zeta)$ is the solution to problem $(\mathcal{P}^T_{s,\zeta})$, we have $\mathbf{I}'_T(\mathbf{u}^s_\zeta) = 0$, which gives:

$$\mathbf{u}^s_\zeta = -MB^*\Phi^s_\zeta,$$

where Φ^s_ζ is the solution of (3.12).

Step 2. Let $(\mathbf{y}^s_\zeta, \mathbf{u}^s_\zeta)$ be the solution to problem $(\mathcal{P}^T_{s,\zeta})$, and let Φ^s_ζ be the solution to Eq. (3.12). From step 1, it follows that $(\mathbf{y}^s_\zeta, \Phi^s_\zeta)$ is a solution of system (3.13). If $(\bar{\mathbf{y}}, \bar{\Phi})$ is a solution of system (3.13), and if we set $\bar{\mathbf{u}} = -MB^*\bar{\Phi}$, with (3.15) we can verify that $\mathbf{I}'_T(\bar{\mathbf{u}}) = 0$, which implies that $\bar{\mathbf{u}} = \mathbf{u}^s_\zeta$. Thus $\bar{\mathbf{y}} = \mathbf{y}^s_\zeta$, and $\bar{\Phi} = \Phi^s_\zeta$, and the second statement in the theorem is established.

Step 3. Let us prove (3.14). From Lemma A.8, it follows that

$$|\Phi^s_\zeta|_{\mathcal{V}^2(\Omega)} \leqslant C\big(\|\Phi^s_\zeta\|_{L^2(s,T;\mathbf{V}^3(\Omega))} + \|\Phi^s_\zeta\|_{H^1(s,T;\mathbf{V}^1_0(\Omega))}\big) \leqslant C\|\mathbf{y}^s_\zeta\|_{L^2(s,T;\mathcal{V}^{-1}(\Omega))},$$

where C does not depend on $s \in [0, T]$. Therefore, for all $\zeta \in \mathbf{V}^0_n(\Omega)$, we have

$$\|\mathbf{y}^s_\zeta\|^2_{L^2(s,T;\mathcal{V}^{-1}(\Omega))} \leqslant C I_T\big(s, \mathbf{y}^s_\zeta, \mathbf{u}^s_\zeta\big) = 2C \int\limits_\Omega \Phi^s_\zeta(s) \cdot \zeta \leqslant C\big|\Phi^s_\zeta(s)\big|_{\mathcal{V}^2(\Omega)} |\zeta|_{\mathcal{V}^{-2}(\Omega)}$$
$$\leqslant C\big(\|\mathbf{y}^s_\zeta\|_{L^2(s,T;\mathcal{V}^{-1}(\Omega))} + |\zeta|_{\mathcal{V}^{-2}(\Omega)}\big)|\zeta|_{\mathcal{V}^{-2}(\Omega)}.$$

Thus, with Young's inequality we obtain

$$\|\mathbf{y}^s_\zeta\|^2_{L^2(s,T;\mathcal{V}^{-1}(\Omega))} \leqslant C I_T\big(s, \mathbf{y}^s_\zeta, \mathbf{u}^s_\zeta\big) \leqslant C|\zeta|^2_{\mathcal{V}^{-2}(\Omega)},$$

for all $\zeta \in \mathbf{V}^0_n(\Omega)$. With the previous estimate for Φ^s_ζ, one has

$$\|\Phi^s_\zeta\|_{L^2(s,T;\mathbf{V}^3(\Omega)) \cap H^1(s,T;\mathbf{V}^1_0(\Omega))} \leqslant C|\zeta|_{\mathcal{V}^{-2}(\Omega)},$$

for all $\zeta \in \mathbf{V}^0_n(\Omega)$. Since $\mathbf{V}^0_n(\Omega)$ is dense in $\mathcal{V}^{-2}(\Omega)$, the proof is complete. □

Corollary 3.2. *For all $s \in [0, T]$, and all $\zeta \in \mathcal{V}^{-2}(\Omega)$, the unique solution $(\mathbf{y}^s_\zeta, \mathbf{u}^s_\zeta)$ to problem $(\mathcal{P}^T_{s,\zeta})$, and the corresponding solution $(\mathbf{y}^s_\zeta, \Phi^s_\zeta)$ to system* (3.13) *obey*

$$I_T\big(s, \mathbf{y}^s_\zeta, \mathbf{u}^s_\zeta\big) = \frac{1}{2}\big\langle \Phi^s_\zeta(s), \zeta\big\rangle_{\mathcal{V}^2(\Omega), \mathcal{V}^{-2}(\Omega)}.$$

Proof. This result is already stated in Corollary 3.1 in the case when ζ belongs to $\mathbf{V}^0_n(\Omega)$. Assume that $\zeta \in \mathcal{V}^{-2}(\Omega)$, and consider a sequence $(\zeta_n)_n \subset \mathbf{V}^0_n(\Omega)$ converging to ζ in $\mathcal{V}^{-2}(\Omega)$. With Theorem 3.4, we have

$$\|\Phi^s_{\zeta_n} - \Phi^s_{\zeta_m}\|_{L^2(s,T;\mathbf{V}^3(\Omega)) \cap H^1(s,T;\mathbf{V}^1_0(\Omega))} \leqslant C|\zeta_n - \zeta_m|^2_{\mathcal{V}^{-2}(\Omega)},$$

for all n and all m. Thus $(\Phi^s_{\zeta_n}(s))_n$ converges to $\Phi^s_\zeta(s)$ in $\mathcal{V}^2(\Omega)$ (because $L^2(s, T; \mathbf{V}^3(\Omega)) \cap H^1(s, T; \mathbf{V}^1_0(\Omega)) \hookrightarrow C([s, T]; \mathcal{V}^2(\Omega))$). We prove the identity satisfied by $I_T(s, \mathbf{y}^s_\zeta, \mathbf{u}^s_\zeta)$ by passing to the limit in the equality

$$I_T\big(s, \mathbf{y}^s_{\zeta_n}, \mathbf{u}^s_{\zeta_n}\big) = \frac{1}{2}\big\langle \Phi^s_{\zeta_n}(s), \zeta_n\big\rangle_{\mathcal{V}^2(\Omega), \mathcal{V}^{-2}(\Omega)}. \qquad \square$$

From Theorem 3.4 and Corollary 3.2 it follows that the operator $\Pi(s)$, defined in (3.6), may be extended to a bounded operator from $\mathcal{V}^{-2}(\Omega)$ into $\mathcal{V}^2(\Omega)$. From the definition of $\Pi(s)$, from Theorem 3.4 and Corollary 3.2, we deduce the following theorem.

Theorem 3.5. *The solution $(\mathbf{y}, \mathbf{u})$ to problem $(\mathcal{P}^T_{0,\mathbf{y}_0})$ belongs to $C([0, T]; \mathcal{V}^{-2}(\Omega)) \times C([0, T]; \mathbf{V}^0(\Gamma))$, it obeys the feedback formula*

$$\mathbf{u}(t) = -MB^*\Pi(t)\mathbf{y}(t),$$

and the optimal cost is given by

$$J(\mathbf{y}, \mathbf{u}) = \frac{1}{2}\big\langle \Pi(0)\mathbf{y}_0, \mathbf{y}_0\big\rangle_{\mathcal{V}^2(\Omega), \mathcal{V}^{-2}(\Omega)} = \frac{1}{2}\big\langle \overline{\Pi}(T)\mathbf{y}_0, \mathbf{y}_0\big\rangle_{\mathcal{V}^2(\Omega), \mathcal{V}^{-2}(\Omega)},$$

where Π (resp. $\overline{\Pi}$) is the solution of (3.7) (*resp.* (3.8)).

Proof. First observe that

$$\|\mathbf{u}\|_{L^2(0,T;\mathbf{V}^{3/2}(\Gamma))} + \|\mathbf{u}\|_{H^{3/4-\varepsilon}(0,T;\mathbf{V}^0(\Gamma))} \leqslant C\|\Phi^0_{\mathbf{y}_0}\|_{L^2(s,T;\mathbf{V}^3(\Omega)) \cap H^1(s,T;\mathbf{V}^1_0(\Omega))} \leqslant C|\mathbf{y}_0|^2_{\mathcal{V}^{-2}(\Omega)},$$

for all $\varepsilon > 0$. Thus $\mathbf{u}$ belongs to $C([0, T]; \mathbf{V}^0(\Gamma))$. The continuity of $\mathbf{y}$ follows from Lemmas A.1 and A.3. The other statements follow from Theorem 3.4 and Corollary 3.2. □

4. Infinite time horizon control problems

In this section we want to study the problem

$$\inf\big\{I(\mathbf{y},\mathbf{u}) \;\big|\; (\mathbf{y},\mathbf{u}) \text{ satisfies (4.1)},\ \mathbf{u}\in \mathbf{V}^{0,0}(\Sigma_\infty)\big\}, \tag{$\mathcal{P}_{0,\mathbf{y}_0}$}$$

where

$$I(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty\int_\Omega \big|(-A_0)^{-1/2}\mathbf{y}\big|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^\infty \big|\mathbf{u}(t)\big|^2_{\mathbf{V}^0(\Gamma)}\,\mathrm{d}t,$$

and

$$\mathbf{y}' = A\mathbf{y} + BM\mathbf{u} \quad \text{in } (0,\infty), \qquad \mathbf{y}(0)=\mathbf{y}_0. \tag{4.1}$$

The main results of this section are stated in Corollaries 4.1 and 4.2 where we highlight the regularizing properties of the solution Π to the algebraic Riccati equation (4.2). The starting point to prove Corollary 4.2 is the estimate stated in Lemma 4.1. This estimate is obtained by passing to the limit in the optimality system obtained in Section 3.2 when the length of the time interval tends to infinity. Since we need precise estimates with irregular initial conditions this framework is not standard, and it is why we have given a complete proof of Lemma 4.1.

4.1. Problem $(\mathcal{P}_{0,\mathbf{y}_0})$ with initial conditions in $\mathbf{V}^0_n(\Omega)$

The analysis of problem $(\mathcal{P}_{0,\mathbf{y}_0})$, in the case when $\mathbf{y}_0\in\mathbf{V}^0_n(\Omega)$, is carried out in [20]. In particular the following result is proved in [20, Theorem 11].

Theorem 4.1. *For all $\mathbf{y}_0\in\mathbf{V}^0_n(\Omega)$, problem $(\mathcal{P}_{0,\mathbf{y}_0})$ admits a unique solution $(\mathbf{y}_{\mathbf{y}_0},\mathbf{u}_{\mathbf{y}_0})$. There exists $\Pi\in\mathcal{L}(\mathbf{V}^0_n(\Omega))$, obeying $\Pi=\Pi^*\geqslant 0$, such that the optimal cost is given by*

$$I(\mathbf{y}_{\mathbf{y}_0},\mathbf{u}_{\mathbf{y}_0}) = \frac{1}{2}(\Pi\mathbf{y}_0,\mathbf{y}_0)_{\mathbf{V}^0_n(\Omega)}.$$

Corollary 4.1. *The operator Π is continuous from $\mathbf{V}^{1/2-\varepsilon}_n(\Omega)$ into $\mathbf{V}^{9/2-\varepsilon}(\Omega)\cap\mathbf{V}^1_0(\Omega)$ for all $0<\varepsilon\leqslant 1/2$.*

The operator $B^*\Pi$ is continuous from $\mathbf{V}^{1/2-\varepsilon}_n(\Omega)$ to $\mathbf{V}^{3-\varepsilon}(\Gamma)$ for all $0<\varepsilon\leqslant 1/2$.

Proof. See [20, Corollary 14]. □

Theorem 4.2. *The unbounded operator $(A_\Pi, D(A_\Pi))$ defined by*:

$$\begin{aligned} &D(A_\Pi) = \big\{\mathbf{y}\in\mathbf{V}^0_n(\Omega) \;\big|\; A\mathbf{y} - BM^2B^*\Pi\mathbf{y}\in\mathbf{V}^0_n(\Omega)\big\},\\ &A_\Pi\mathbf{y} = A\mathbf{y} - BM^2B^*\Pi\mathbf{y}, \end{aligned}$$

is the infinitesimal generator of an exponentially stable semigroup on $\mathbf{V}^0_n(\Omega)$.

The operator Π is the unique weak solution to the algebraic Riccati equation,

$$\begin{aligned} &\Pi^* = \Pi\in\mathcal{L}\big(\mathbf{V}^0_n(\Omega)\big) \quad \textit{and} \quad \Pi\geqslant 0,\\ &\textit{for all } \mathbf{y}\in\mathbf{V}^0_n(\Omega),\quad \Pi\mathbf{y}\in\mathbf{V}^2(\Omega)\cap\mathbf{V}^1_0(\Omega) \quad \textit{and} \quad |\Pi\mathbf{y}|_{\mathbf{V}^2(\Omega)}\leqslant C|\mathbf{y}|_{\mathbf{V}^0_n(\Omega)},\\ &A^*\Pi + \Pi A - \Pi BM^2B^*\Pi + (-A_0)^{-1} = 0. \end{aligned} \tag{4.2}$$

Proof. See [20, Theorem 15]. Notice in particular that, due to Theorem 2.1, the pair $(A,(-A_0)^{-1/2})$ is exponentially detectable. □

4.2. Problem $(\mathcal{P}_{0,\mathbf{y}_0})$ with initial conditions in $\mathcal{V}^{-2}(\Omega)$

As explained in Section 3.2, we have to study problem $(\mathcal{P}_{0,\mathbf{y}_0})$ when $\mathbf{y}_0 \in \mathcal{V}^{-2}(\Omega)$.

Theorem 4.3. *For all $\mathbf{y}_0 \in \mathcal{V}^{-2}(\Omega)$, problem $(\mathcal{P}_{0,\mathbf{y}_0})$ admits a unique solution $(\mathbf{y}_{\mathbf{y}_0}, \mathbf{u}_{\mathbf{y}_0})$. The operator $\Pi \in \mathcal{L}(\mathbf{V}^0_n(\Omega), \mathbf{V}^4(\Omega) \cap \mathbf{V}^1_0(\Omega))$ in Theorem 4.1 may be extended to a bounded linear operator from $\mathcal{V}^{-2}(\Omega)$ into $\mathcal{V}^2(\Omega)$, and the optimal cost of problem $(\mathcal{P}_{0,\mathbf{y}_0})$ obeys*

$$I(\mathbf{y}_{\mathbf{y}_0}, \mathbf{u}_{\mathbf{y}_0}) = \frac{1}{2}\langle \Pi \mathbf{y}_0, \mathbf{y}_0\rangle_{\mathcal{V}^2(\Omega),\mathcal{V}^{-2}(\Omega)}.$$

Proof. *Step 1.* The existence of $\mathbf{u} \in L^2(0,\infty; \mathbf{V}^0(\Gamma))$ such that $I(\mathbf{y}_\mathbf{u}, \mathbf{u}) < \infty$, where $\mathbf{y}_\mathbf{u}$ is the solution of Eq. (4.1) corresponding to $\mathbf{u}$, may be deduced from Theorem 4.1. Indeed let us denote by $\mathbf{z}^0$ the solution of Eq. (4.1) corresponding to $\mathbf{u} = 0$. Notice that $\mathbf{z}^0(T) \in \mathbf{V}^0_n(\Omega)$ for any $T > 0$. Let us fix $T > 0$, and let $(\mathbf{y}_{\mathbf{z}^0(T)}, \mathbf{u}_{\mathbf{z}^0(T)})$ be the solution to problem $(\mathcal{P}_{0,\mathbf{z}^0(T)})$. Now we set

$$\mathbf{v}(t) = \begin{cases} 0 & \text{if } t \in (0,T), \\ \mathbf{u}_{\mathbf{z}^0(T)}(t-T) & \text{if } t > T. \end{cases}$$

Since $\mathbf{y}_\mathbf{v}$, the solution of Eq. (4.1) corresponding to $\mathbf{u} = \mathbf{v}$, obeys $\mathbf{y}_\mathbf{v}(t) = \mathbf{y}_{\mathbf{z}^0(T)}(t-T)$ for all $t > T$, we easily verify that $I(\mathbf{y}_\mathbf{v}, \mathbf{v}) < \infty$. Now, the existence of a unique solution $(\mathbf{y}_{\mathbf{y}_0}, \mathbf{u}_{\mathbf{y}_0})$ to $(\mathcal{P}_{0,\mathbf{y}_0})$ follows from classical arguments (see [17, proof of Theorem 2.3.3.1 (i), p. 135]).

Let $\overline{\Pi}$ be the solution to Eq. (3.8). From the dynamic programming principle, it follows that the mapping,

$$k \mapsto \langle \overline{\Pi}(k)\mathbf{y}_0, \mathbf{y}_0\rangle_{\mathcal{V}^2(\Omega),\mathcal{V}^{-2}(\Omega)},$$

is nondecreasing, and we have

$$\frac{1}{2}\langle \overline{\Pi}(k)\mathbf{y}_0, \mathbf{y}_0\rangle_{\mathcal{V}^2(\Omega),\mathcal{V}^{-2}(\Omega)} \leqslant I(\mathbf{y}_{\mathbf{y}_0}, \mathbf{u}_{\mathbf{y}_0}) < \infty.$$

As in [8], or in [17], we can show that there exists an operator $\Pi_\infty \in \mathcal{L}(\mathcal{V}^{-2}(\Omega), \mathcal{V}^2(\Omega))$, satisfying $\Pi_\infty = \Pi_\infty^* \geqslant 0$, and

$$\Pi_\infty \mathbf{y}_0 = \lim_{k\to\infty} \overline{\Pi}(k)\mathbf{y}_0 \quad \text{for all } \mathbf{y}_0 \in \mathcal{V}^{-2}(\Omega).$$

It is clear that the restriction of Π_∞ to $\mathbf{V}^0_n(\Omega)$ is identical to Π. Thus, Π_∞ is the continuous extension of Π to $\mathcal{V}^{-2}(\Omega)$. For notational simplicity, we still denote this extension by Π.

Step 2. Let us show that $I(\mathbf{y}_{\mathbf{y}_0}, \mathbf{u}_{\mathbf{y}_0}) = \frac{1}{2}\langle \Pi \mathbf{y}_0, \mathbf{y}_0\rangle_{\mathcal{V}^2(\Omega),\mathcal{V}^{-2}(\Omega)}$. Problem $(\mathcal{P}^k_{0,\mathbf{y}_0})$ admits a unique solution $(\mathbf{y}_k, \mathbf{u}_k)$ characterized by

$$\begin{aligned} &\mathbf{y}_k' = A\mathbf{y}_k + BM\mathbf{u}_k \quad \text{in } (0,k), \quad \mathbf{y}_k(0) = \mathbf{y}_0, \\ &-\Phi_k' = A^*\Phi_k + (-A_0)^{-1}\mathbf{y}_k \quad \text{in } (0,k), \quad \Phi_k(k) = 0, \\ &\mathbf{u}_k = -MB^*\Phi_k. \end{aligned} \tag{4.3}$$

Let us denote by $\tilde{\mathbf{u}}_k$ the extension by zero of $\mathbf{u}_k$ to (k,∞), and by $\tilde{\mathbf{y}}_k$ the extension by zero of $\mathbf{y}_k$ to (k,∞). Since we have

$$\begin{aligned} &\int_0^k \int_\Omega \left|(-A_0)^{-1/2}\mathbf{y}_k\right|^2 \mathrm{d}x\,\mathrm{d}t + \int_0^k \left|\mathbf{u}_k(t)\right|^2_{\mathbf{V}^0(\Gamma)}\mathrm{d}t \\ &\quad \leqslant \int_0^\infty \int_\Omega \left|(-A_0)^{-1/2}\mathbf{y}_{\mathbf{y}_0}\right|^2 \mathrm{d}x\,\mathrm{d}t + \int_0^\infty \left|\mathbf{u}_{\mathbf{y}_0}(t)\right|^2_{\mathbf{V}^0(\Gamma)}\mathrm{d}t, \end{aligned}$$

the sequences $(\tilde{\mathbf{y}}_k)_k$ and $(\tilde{\mathbf{u}}_k)_k$ are bounded respectively in $L^2(0,\infty; \mathcal{V}^{-1}(\Omega))$ and $L^2(0,\infty; \mathbf{V}^0(\Gamma))$. Thus, there exist $\mathbf{y}_\infty \in L^2(0,\infty; \mathcal{V}^{-1}(\Omega))$ and $\mathbf{u}_\infty \in L^2(0,\infty; \mathbf{V}^0(\Gamma))$ such that

$$\tilde{\mathbf{u}}_k \rightharpoonup \mathbf{u}_\infty \quad \text{weakly in } L^2\big(0,\infty;\mathbf{V}^0(\Gamma)\big),$$
$$\tilde{\mathbf{y}}_k \rightharpoonup \mathbf{y}_\infty \quad \text{weakly in } L^2\big(0,\infty;\mathcal{V}^{-1}(\Omega)\big).$$

By passing to the lower limit in the above inequality we obtain

$$\int_0^\infty\int_\Omega \big|(-A_0)^{-1/2}\mathbf{y}_\infty\big|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \big|\mathbf{u}_\infty(t)\big|^2_{\mathbf{V}^0(\Gamma)}\,\mathrm{d}t$$
$$\leqslant \int_0^\infty\int_\Omega \big|(-A_0)^{-1/2}\mathbf{y}_{\mathbf{y}_0}\big|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \big|\mathbf{u}_{\mathbf{y}_0}(t)\big|^2_{\mathbf{V}^0(\Gamma)}\,\mathrm{d}t.$$

And by passing to the limit in the equation satisfied by $(\mathbf{y}_k,\mathbf{u}_k)$, we have

$$\mathbf{y}'_\infty = A\mathbf{y}_\infty + BM\mathbf{u}_\infty \quad \text{in } (0,\infty), \qquad \mathbf{y}_\infty(0) = \mathbf{y}_0.$$

Thus, the pair $(\mathbf{y}_\infty,\mathbf{u}_\infty)$ is admissible for $(\mathcal{P}_{0,\mathbf{y}_0})$ and we have

$$(\mathbf{y}_\infty,\mathbf{u}_\infty) = (\mathbf{y}_{\mathbf{y}_0},\mathbf{u}_{\mathbf{y}_0}),$$

because $I(\mathbf{y}_\infty,\mathbf{u}_\infty) \leqslant I(\mathbf{y}_{\mathbf{y}_0},\mathbf{u}_{\mathbf{y}_0})$. Therefore we can claim that

$$\tilde{\mathbf{u}}_k \to \mathbf{u}_{\mathbf{y}_0} \quad \text{in } L^2\big(0,\infty;\mathbf{V}^0(\Gamma)\big) \quad \text{and} \quad \tilde{\mathbf{y}}_k \to \mathbf{y}_{\mathbf{y}_0} \quad \text{in } L^2\big(0,\infty;\mathcal{V}^{-1}(\Omega)\big).$$

Since

$$I_k(0,\mathbf{y}_k,\mathbf{u}_k) = \frac{1}{2}\big\langle\overline{\Pi}(k)\mathbf{y}_0,\mathbf{y}_0\big\rangle_{\mathcal{V}^2(\Omega),\mathcal{V}^{-2}(\Omega)},$$

by passing to the limit when k tends to infinity, we obtain

$$I(\mathbf{y}_{\mathbf{y}_0},\mathbf{u}_{\mathbf{y}_0}) = \frac{1}{2}\langle\Pi\mathbf{y}_0,\mathbf{y}_0\rangle_{\mathcal{V}^2(\Omega),\mathcal{V}^{-2}(\Omega)}. \qquad \square$$

We denote by $\varphi(\mathbf{y}_0)$ the value function of problem $(\mathcal{P}_{0,\mathbf{y}_0})$, that is:

$$\varphi(\mathbf{y}_0) = I(\mathbf{y}_{\mathbf{y}_0},\mathbf{u}_{\mathbf{y}_0}).$$

Lemma 4.1. *For every $\mathbf{y}_0 \in \mathcal{V}^{-2}(\Omega)$, the system*

$$\begin{aligned} &\mathbf{y}' = A\mathbf{y} - BM^2B^*\Phi \quad \textit{in } (0,\infty), \qquad \mathbf{y}(0) = \mathbf{y}_0,\\ &-\Phi' = A^*\Phi + (-A_0)^{-1}\mathbf{y} \quad \textit{in } (0,\infty), \qquad \Phi(\infty) = 0,\\ &\Phi(t) = \Pi\mathbf{y}(t) \quad \textit{for all } t \in (0,\infty), \end{aligned} \tag{4.4}$$

admits a unique solution in $L^2(0,\infty;\mathcal{V}^{-1}(\Omega)) \times \mathbf{V}^{2,1}(Q_\infty)$. This solution belongs to $(L^2(0,\infty;\mathcal{V}^{-1}(\Omega)) \cap H^1(0,\infty;\mathcal{V}^{-3}(\Omega))) \times (L^2(0,\infty;\mathbf{V}^3(\Omega)) \cap H^1(0,\infty;\mathbf{V}^1_0(\Omega)))$ and it satisfies:

$$\|\mathbf{y}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))\cap H^1(0,\infty;\mathcal{V}^{-3}(\Omega))} + \|\Phi\|_{L^2(0,\infty;\mathbf{V}^3(\Omega))\cap H^1(0,\infty;\mathbf{V}^1_0(\Omega))} \leqslant C|\mathbf{y}_0|_{\mathcal{V}^{-2}(\Omega)}.$$

The pair $(\mathbf{y},-MB^*\Phi)$ is the solution of $(\mathcal{P}_{0,\mathbf{y}_0})$.

Proof. The proof of this lemma is very similar to that of [19, Lemma 4.2], where only the case $\mathbf{y}_0 \in \mathbf{V}^0_n(\Omega)$ is considered. Since the analysis with $\mathbf{y}_0 \in \mathcal{V}^{-2}(\Omega)$ is more delicate, we completely rewrite the proof for the convenience of the reader. For notational simplicity the solution to $(\mathcal{P}_{0,\mathbf{y}_0})$ will now be denoted by $(\hat{\mathbf{y}},\hat{\mathbf{u}})$. We denote by $\varphi_k(0,\mathbf{y}_0)$ the value function of problem $(\mathcal{P}^k_{0,\mathbf{y}_0})$ and by $\varphi_k(\bar{t},\zeta)$ the value function of problem $(\mathcal{P}^k_{\bar{t},\zeta})$.

Step 1. Let $(\mathbf{y}^{\bar{t}}_k,\mathbf{u}^{\bar{t}}_k)$ be the solution of $(\mathcal{P}^k_{\bar{t},\mathbf{y}_k(\bar{t})})$, and let $(\mathbf{y}_k,\mathbf{u}_k)$ be the solution of $(\mathcal{P}^k_{0,\mathbf{y}_0})$ characterized by (4.3). Denote by $\Phi^{\bar{t}}_k$ the adjoint state corresponding to $(\mathbf{y}^{\bar{t}}_k,\mathbf{u}^{\bar{t}}_k)$, and by Φ_k the adjoint state corresponding to $(\mathbf{y}_k,\mathbf{u}_k)$. From the dynamic programming principle it follows that $(\mathbf{y}_k^{\bar t}, \mathbf{u}_k^{\bar t}, \Phi_k^{\bar t})(t) = (\mathbf{y}_k, \mathbf{u}_k, \Phi_k)(t)$ for all $t \in (\bar t, k)$. Therefore we have $\Phi_k^{\bar t}(\bar t) = \Phi_k(\bar t) \in \partial_y \varphi_k(\bar t, \mathbf{y}_k(\bar t))$, that is $\Phi_k(\bar t) = \overline{\Pi}(k - \bar t)\mathbf{y}_k(\bar t)$.

Let $\tilde{\mathbf{y}}_k$ be the extension by zero of $\mathbf{y}_k$ to (k, ∞). In the proof of Theorem 4.3, we have shown that $(\tilde{\mathbf{y}}_k)_k$ is bounded in $L^2(0, \infty; \mathcal{V}^{-1}(\Omega))$ and that it converges to $\hat{\mathbf{y}}$ in $L^2(0, \infty; \mathcal{V}^{-1}(\Omega))$. Thus

$$\big|\Phi_k(\bar t)\big|_{\mathcal{V}^2(\Omega)} \leqslant C\big\|\overline{\Pi}(k - \bar t)\big\|_{\mathcal{L}(\mathcal{V}^{-2}(\Omega), \mathcal{V}^2(\Omega))}\big|\mathbf{y}_k(\bar t)\big|_{\mathcal{V}^{-1}(\Omega)} \leqslant C\big|\mathbf{y}_k(\bar t)\big|_{\mathcal{V}^{-1}(\Omega)},$$

and

$$\|\tilde\Phi_k\|_{L^2(0,\infty;\mathcal{V}^2(\Omega))} \leqslant C\|\tilde{\mathbf{y}}_k\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))},$$

where $\tilde\Phi_k$ is the extension by zero of Φ_k to (k, ∞). Therefore $(\tilde\Phi_k)_k$ is bounded in $L^2(0, \infty; \mathcal{V}^2(\Omega))$. Observe that $\tilde\Phi_k$ is also the solution of the equation

$$-\tilde\Phi_k' = \big(A^* - \lambda_0 I\big)\tilde\Phi_k + (-A_0)^{-1}\tilde{\mathbf{y}}_k + \lambda_0\tilde\Phi_k, \quad \tilde\Phi_k(\infty) = 0.$$

Due to Lemma A.5, $(\tilde\Phi_k)_k$ is also bounded in $L^2(0, \infty; \mathbf{V}^3(\Omega)) \cap H^1(0, \infty; \mathbf{V}_0^1(\Omega))$. From Young's inequality for convolutions it follows that $(\tilde\Phi_k)_k$ is also bounded in $L^\infty(0, \infty; \mathcal{V}^2(\Omega))$. There then exists $\hat\Phi \in L^2(0, \infty; \mathbf{V}^3(\Omega)) \cap H^1(0, \infty; \mathbf{V}_0^1(\Omega))$ such that, after extraction of a subsequence, we have:

$$\begin{aligned}&\tilde\Phi_k \rightharpoonup \hat\Phi \quad \text{weakly in } L^2\big(0, \infty; \mathbf{V}^3(\Omega)\big) \cap H^1\big(0, \infty; \mathbf{V}_0^1(\Omega)\big), \quad \text{and}\\ &\|\hat\Phi\|_{L^2(0,\infty;\mathbf{V}^3(\Omega))\cap H^1(0,\infty;\mathbf{V}_0^1(\Omega))} \leqslant C\|\hat{\mathbf{y}}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}.\end{aligned}$$

Moreover, $\hat\Phi$ obeys the equation

$$\hat\Phi(t) = \int_t^\infty \mathrm{e}^{(A^*-\lambda_0 I)(\tau - t)}\big((-A_0)^{-1}\hat{\mathbf{y}}(\tau) + \lambda_0\hat\Phi(\tau)\big)\,\mathrm{d}\tau \quad \text{for all } t \geqslant 0.$$

Thus,

$$\tilde\Phi_k(t) \rightharpoonup \hat\Phi(t) \quad \text{weakly in } \mathcal{V}^2(\Omega) \text{ for all } t \geqslant 0.$$

Step 2. Since $(\tilde\Phi_k)_k$ converges to $\hat\Phi$, weakly in $L^2(0, \infty; \mathcal{V}^2(\Omega))$, from Proposition 2.1, it follows that the sequence $(\tilde{\mathbf{u}}_k)_k = (-MB^*\tilde\Phi_k)_k$ converges weakly in $L^2(0, \infty; \mathbf{V}^0(\Gamma))$ to $-MB^*\hat\Phi$. Thus $\hat{\mathbf{u}} = -MB^*\hat\Phi$, and the pair $(\hat{\mathbf{y}}, \hat\Phi)$ obeys the first two equations in (4.4).

Step 3. Let us show that if $(\mathbf{y}, \Phi) \in L^2(0, \infty; \mathcal{V}^{-1}(\Omega)) \times \mathbf{V}^{2,1}(Q_\infty)$ is a solution of the first two equations of system (4.4), then $\mathbf{y}$ belongs $L^2(0, \infty; \mathcal{V}^{-1}(\Omega)) \cap H^1(0, \infty; \mathcal{V}^{-3}(\Omega))$, $\Phi \in L^2(0, \infty; \mathbf{V}^3(\Omega)) \cap H^1(0, \infty; \mathbf{V}_0^1(\Omega))$, and

$$\begin{aligned}&\|\mathbf{y}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))\cap H^1(0,\infty;\mathcal{V}^{-3}(\Omega))} + \|\Phi\|_{L^2(0,\infty;\mathbf{V}^3(\Omega))\cap H^1(0,\infty;\mathbf{V}_0^1(\Omega))}\\ &\quad\leqslant C\big(|\mathbf{y}_0|_{\mathcal{V}^{-2}(\Omega)} + \|\Phi\|_{L^2(0,\infty;\mathbf{V}_n^0(\Omega))}\big).\end{aligned} \tag{4.5}$$

We rewrite the first two equations of system (4.4) as follows

$$\begin{aligned}&\mathbf{y}' = \big(A - \lambda_0(-A_0)^{-1/2}\big)\mathbf{y} - BM^2B^*\Phi + \lambda_0(-A_0)^{-1/2}\mathbf{y} \quad \text{in } (0, \infty), \quad \mathbf{y}(0) = \mathbf{y}_0,\\ &-\Phi' = \big(A^* - \lambda_0 I\big)\Phi + (-A_0)^{-1}\mathbf{y} + \lambda_0\Phi \quad \text{in } (0, \infty), \quad \Phi(\infty) = 0.\end{aligned} \tag{4.6}$$

We set

$$\begin{aligned}&\mathbf{y}_i(t) = \mathrm{e}^{t(A-\lambda_0(-A_0)^{-1/2})}\mathbf{y}_0 \quad \text{and}\\ &\mathbf{y}_b(t) = \int_0^t \mathrm{e}^{(t-\tau)(A-\lambda_0(-A_0)^{-1/2})}\big(\lambda_0(-A_0)^{-1/2}\mathbf{y}(\tau) - BM^2B^*\Phi(\tau)\big)\,\mathrm{d}\tau.\end{aligned}$$

From Lemma A.5, it yields

$$\|\Phi\|_{\mathbf{V}^{2,1}(Q_\infty)} \leqslant C\big(\|\mathbf{y}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))} + \|\Phi\|_{L^2(0,\infty;\mathbf{V}_n^0(\Omega))}\big).$$

Applying another time Lemma A.5, we have

$$\begin{aligned}\|\Phi\|_{L^2(0,\infty;\mathbf{V}^3(\Omega))\cap H^1(0,\infty;\mathbf{V}^1_0(\Omega))} &\leqslant C\|\mathbf{y}+\lambda_0\Phi\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}\\ &\leqslant C\big(\|\mathbf{y}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}+\|\Phi\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))}\big).\end{aligned}$$

Since

$$\|\mathbf{y}\|^2_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}\leqslant C\langle\Pi\mathbf{y}_0,\mathbf{y}_0\rangle_{\mathcal{V}^2(\Omega),\mathcal{V}^{-2}(\Omega)}\leqslant C|\mathbf{y}_0|^2_{\mathcal{V}^{-2}(\Omega)},$$

we obtain

$$\|\Phi\|_{L^2(0,\infty;\mathbf{V}^3(\Omega))\cap H^1(0,\infty;\mathbf{V}^1_0(\Omega))}\leqslant C\big(|\mathbf{y}_0|_{\mathcal{V}^{-2}(\Omega)}+\|\Phi\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))}\big). \tag{4.7}$$

We know that

$$\|\mathbf{y}_i\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))\cap H^1(0,\infty;\mathcal{V}^{-3}(\Omega))}\leqslant C|\mathbf{y}_0|_{\mathcal{V}^{-2}(\Omega)}.$$

With Proposition 2.1, we have

$$\big\|B^*\Phi\big\|_{L^2(0,\infty;\mathbf{V}^{3/2}(\Gamma))\cap H^{3/4-\varepsilon/2}(0,\infty;\mathbf{V}^{\varepsilon}(\Gamma))}\leqslant C\|\Phi\|_{L^2(0,\infty;\mathbf{V}^3(\Omega))\cap H^1(0,\infty;\mathbf{V}^1_0(\Omega))},$$

for all $\varepsilon>0$. Applying Lemma A.3, we obtain:

$$\|\mathbf{y}_b\|_{\mathbf{V}^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_\infty)}\leqslant C\big(\|\mathbf{y}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}+\big\|B^*\Phi\big\|_{\mathbf{V}^{1,1/2}(\Sigma_\infty)}\big)\quad\text{for all }\varepsilon>0.$$

Due to the previous estimate for Φ, one has

$$\|\mathbf{y}_b\|_{\mathbf{V}^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_\infty)}\leqslant C\big(|\mathbf{y}_0|_{\mathcal{V}^{-2}(\Omega)}+\|\mathbf{\Phi}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))}\big)\quad\text{for all }\varepsilon>0.$$

From the equation

$$\mathbf{y}_b'=\big(A-\lambda_0(-A_0)^{-1/2}\big)\mathbf{y}_b-BM^2B^*\Phi+\lambda_0(-A_0)^{-1/2}\mathbf{y}\quad\text{in }(0,\infty),\qquad\mathbf{y}(0)=0,$$

with (4.7), we deduce the following estimate of $\mathbf{y}_b$ in $H^1(0,\infty;\mathcal{V}^{-3}(\Omega))$:

$$\begin{aligned}\|\mathbf{y}_b\|_{H^1(0,\infty;\mathcal{V}^{-3}(\Omega))} &\leqslant C\Big(\big\|\big(A-\lambda_0(-A_0)^{-1/2}\big)\mathbf{y}_b-BM^2B^*\Phi+\lambda_0(-A_0)^{-1/2}\mathbf{y}\big\|_{L^2(0,\infty;\mathcal{V}^{-3}(\Omega))}\\ &\quad+\|\mathbf{y}_b\|_{L^2(0,\infty;\mathcal{V}^{-3}(\Omega))}\Big)\\ &\leqslant C\big(\|\mathbf{y}_b\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}+\big\|B^*\Phi\big\|_{L^2(0,\infty;\mathbf{V}^0(\Gamma))}\big)\\ &\leqslant C\big(\|\mathbf{y}_b\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}+\|\Phi\|_{L^2(0,\infty;\mathbf{V}^3(\Omega))}\big)\\ &\leqslant C\big(|\mathbf{y}_0|_{\mathcal{V}^{-2}(\Omega)}+\|\Phi\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))}\big).\end{aligned}$$

Estimate (4.5) is proved.

Step 4. We show that the pair $(\hat{\mathbf{y}},\hat{\Phi})$ *obeys the third equation of system* (4.4). We know that

$$\tilde{\Phi}_k(t)\rightharpoonup\hat{\Phi}(t)\quad\text{weakly in }\mathcal{V}^2(\Omega)\text{ for all }t\geqslant 0.$$

Since

$$\Phi_k(t)\in\partial_{\mathbf{y}}\varphi_k\big(t,\mathbf{y}_k(t)\big),\qquad\Phi_k(t)\rightharpoonup\hat{\Phi}(t)\quad\text{weakly in }\mathcal{V}^2(\Omega),$$

and

$$\varphi_k\big(t,\mathbf{y}_k(t)\big)\to\varphi\big(\hat{\mathbf{y}}(t)\big)\quad\text{as }k\to\infty,$$

we deduce that

$$\hat{\Phi}(t)\in\partial\varphi\big(\hat{\mathbf{y}}(t)\big),\qquad\text{i.e. }\hat{\Phi}(t)=\Pi\hat{\mathbf{y}}(t).$$

Thus we have shown that $\hat{\Phi}$ is the solution of the second and the third equation in (4.4) corresponding to $\hat{\mathbf{y}}$.

Step 5. Uniqueness. If a solution $(\mathbf{y},\Phi)\in L^2(0,\infty;\mathcal{V}^{-1}(\Omega))\times\mathbf{V}^{2,1}(Q_\infty)$ to system (4.4), due to step 3, it obeys (4.5), and we can show that

$$\int_0^k \left|(-A_0)^{-1/2}\mathbf{y}(t)\right|^2_{\mathbf{V}^0_n(\Omega)}\,\mathrm{d}t + \int_0^k \left|MB^*\Phi(t)\right|^2_{\mathbf{V}^0(\Gamma)}\,\mathrm{d}t$$
$$= \langle \mathbf{y}_0, \Phi(0)\rangle_{\mathcal{V}^{-2}(\Omega),\mathcal{V}^2(\Omega)} - \langle \mathbf{y}(k), \Phi(k)\rangle_{\mathcal{V}^{-2}(\Omega),\mathcal{V}^2(\Omega)}.$$

Passing to the limit when k tends to infinity we obtain:

$$\int_0^\infty \left|(-A_0)^{-1/2}\mathbf{y}(t)\right|^2_{\mathbf{V}^0_n(\Omega)}\,\mathrm{d}t + \int_0^\infty \left|MB^*\Phi(t)\right|^2_{\mathbf{V}^0(\Gamma)}\,\mathrm{d}t = \langle \mathbf{y}_0, \Phi(0)\rangle_{\mathcal{V}^{-2}(\Omega),\mathcal{V}^2(\Omega)}.$$

Thus if $\mathbf{y}_0 = 0$ we have $\mathbf{y} = 0$. From the relation $\Phi = \Pi\mathbf{y}$ we deduce that $\Phi = 0$, and the uniqueness is established.

Step 6. Final estimate. From the previous steps it follows that $(\hat{\mathbf{y}}, \hat{\Phi})$ is the unique solution to system (4.4). Since $\|\hat{\mathbf{y}}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))} \leqslant C|\mathbf{y}_0|_{\mathcal{V}^{-2}(\Omega)}$, we have

$$\|\hat{\Phi}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))} \leqslant C|\mathbf{y}_0|_{\mathcal{V}^{-2}(\Omega)},$$

the estimate of the lemma follows from (4.5). □

Corollary 4.2. *If* $\mathbf{y}_0 \in \mathbf{V}_n^{1/2-\varepsilon}(\Omega)$ *for some* $0 < \varepsilon \leqslant 1/2$, *then the solution* $(\mathbf{y}, \Phi)$ *of system* (4.4) *belongs to* $\mathbf{V}^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_\infty) \times (L^2(0,\infty;\mathbf{V}^{11/2-\varepsilon}(\Omega)) \cap H^{7/4-\varepsilon/2}(0,\infty;\mathbf{V}^2(\Omega)))$, *and we have*:

$$\|\mathbf{y}\|_{\mathbf{V}^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_\infty)} + \|\Phi\|_{L^2(0,\infty;\mathbf{V}^{11/2-\varepsilon}(\Omega))\cap H^{7/4-\varepsilon/2}(0,\infty;\mathbf{V}^2(\Omega))} \leqslant C|\mathbf{y}_0|_{\mathbf{V}_n^{1/2-\varepsilon}(\Omega)},$$
$$\left\|B^*\Phi\right\|_{L^2(0,\infty;\mathbf{V}^{4-\varepsilon}(\Gamma))\cap H^{7/4-\varepsilon/2}(0,\infty;\mathbf{V}^{1/2}(\Gamma))} \leqslant C|\mathbf{y}_0|_{\mathbf{V}_n^{1/2-\varepsilon}(\Omega)}.$$

Proof. See [20, Corollary 13]. □

5. A time-dependent feedback operator

In order to improve the regularity of the optimal state $\mathbf{y}_{\mathbf{y}_0}$ of problem $(P_{0,\mathbf{y}_0})$, we modify the control operator in the state equation. We introduce a weight function $\theta \in C^\infty([0,\infty))$ satisfying

$$\theta(t) \in [0,1] \text{ for all } t \in \mathbb{R}^+, \quad \theta(0) = 0 \quad \text{and} \quad \theta(t) = 1 \text{ for all } t \geqslant t_0,$$

where $t_0 > 0$ is given fixed. The new state equation is

$$\mathbf{y}' = A\mathbf{y} + \theta BM\mathbf{u} \quad \text{in } (0,\infty), \quad \mathbf{y}(0) = \mathbf{y}_0. \tag{5.1}$$

Due to Lemmas A.1 and A.2, and Remark A.1, if $\mathbf{y}_0 \in \mathbf{V}_0^{1/2+\varepsilon}(\Omega)$ and if $M\mathbf{u} \in \mathbf{V}^{1,1/2}(\Sigma_\infty)$ for some $\varepsilon > 0$, then the solution to Eq. (5.1) belongs to $L^2_{\mathrm{loc}}([0,\infty);\mathbf{V}^{3/2+\varepsilon}(\Omega)) \cap H^{3/4+\varepsilon/2}_{\mathrm{loc}}([0,\infty);\mathbf{V}^0_n(\Omega))$. This kind of regularity is necessary to deal with the stabilization of Navier–Stokes equations in three dimension.

In this section, we want to study, in function of the regularity of initial condition, the regularity of solutions of the control problem

$$\inf\left\{I(s,\mathbf{y},\mathbf{u}) \mid (\mathbf{y},\mathbf{u}) \text{ satisfies (5.2)},\ \mathbf{u} \in \mathbf{V}^{0,0}(\Sigma_{s,\infty})\right\}, \tag*{$(\mathcal{Q}_{s,\mathbf{y}_0})$}$$

where

$$I(s,\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_s^\infty\int_\Omega \left|(-A_0)^{-1/2}\mathbf{y}\right|^2 + \frac{1}{2}\int_s^\infty\int_\Gamma |\mathbf{u}|^2,$$

and

$$\mathbf{y}' = A\mathbf{y} + \theta BM\mathbf{u} \quad \text{in } (s,\infty), \quad \mathbf{y}(s) = \mathbf{y}_0. \tag{5.2}$$

5.1. Problem $(\mathcal{Q}_{s,\mathbf{y}_0})$ with initial conditions in $\mathbf{V}^0_n(\Omega)$

Theorem 5.1. *For all $\mathbf{y}_0 \in \mathbf{V}^0_n(\Omega)$, and all $s \in [0,\infty)$, problem $(\mathcal{Q}_{s,\mathbf{y}_0})$ admits a unique solution $(\mathbf{y}^s_{\mathbf{y}_0}, \mathbf{u}^s_{\mathbf{y}_0})$. There exists $\Pi(s) \in \mathcal{L}(\mathbf{V}^0_n(\Omega))$ such that the optimal cost is given by*

$$I\big(s, \mathbf{y}^s_{\mathbf{y}_0}, \mathbf{u}^s_{\mathbf{y}_0}\big) = \frac{1}{2}\big(\Pi(s)\mathbf{y}_0, \mathbf{y}_0\big)_{\mathbf{V}^0_n(\Omega)}.$$

Proof. The proof is analogous to the one of Theorem 4.1. □

Lemma 5.1. *For every $\mathbf{y}_0 \in \mathbf{V}^0_n(\Omega)$, the system*

$$\begin{aligned} &\mathbf{y}' = A\mathbf{y} - \theta^2 BM^2B^*\Phi \quad \text{in } (s,\infty), \quad \mathbf{y}(s) = \mathbf{y}_0, \\ &-\Phi' = A^*\Phi + (-A_0)^{-1}\mathbf{y} \quad \text{in } (s,\infty), \quad \Phi(\infty) = 0, \\ &\Phi(t) = \Pi(t)\mathbf{y}(t) \quad \text{for all } t \in (s,\infty), \end{aligned} \tag{5.3}$$

admits a unique solution in $L^2(s,\infty;\mathbf{V}^0_n(\Omega)) \times \mathbf{V}^{2,1}(Q_{s,\infty})$. This solution belongs to $C_b([s,\infty);\mathbf{V}^0_n(\Omega)) \cap \mathbf{V}^{1,1/2}(Q_{s,\infty}) \times (L^2(s,\infty;\mathbf{V}^5(\Omega)) \cap H^{3/2}(s,\infty;\mathbf{V}^2(\Omega)))$ and it satisfies:

$$\|\mathbf{y}\|_{C_b([s,\infty);\mathbf{V}^0_n(\Omega))} + \|\mathbf{y}\|_{\mathbf{V}^{1,1/2}(Q_{s,\infty})} + \|\Phi\|_{L^2(s,\infty;\mathbf{V}^5(\Omega))\cap H^{3/2}(s,\infty;\mathbf{V}^2(\Omega))} \leqslant C|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}. \tag{5.4}$$

The pair $(\mathbf{y}, -\theta MB^*\Phi)$ is the solution of $(\mathcal{Q}_{s,\mathbf{y}_0})$.

Proof. The proof is completely analogous to the one of Lemma 4.1. The regularity of Φ follows from Lemma A.5. □

Remark 5.1. In the first equation of system (5.3), the operator A is considered as the infinitesimal generator of an analytic semigroup on $\mathcal{V}^{-2}(\Omega)$, with domain $\mathbf{V}^0_n(\Omega)$. Using the first equation in (5.3), the fact that $\mathbf{y} \in \mathbf{V}^{1,1/2}(Q_{s,\infty})$, and estimate (5.4), we can prove that

$$\|\mathbf{y}\|_{H^1(s,\infty;\mathcal{V}^{-1}(\Omega))} \leqslant C\|\mathbf{y}\|_{L^2(s,\infty;\mathbf{V}^1(\Omega))} \leqslant C|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}.$$

Since $\mathbf{y}_0 \in \mathbf{V}^0_n(\Omega)$ and $\theta^2BM^2B^*\Phi \in H^1(s,\infty;\mathcal{V}^{-2}(\Omega))$, with estimate (5.4) and the fact that the semigroup generated by $A - \lambda_0 I$ is exponentially stable on $\mathcal{V}^{-2}(\Omega)$, we have

$$\begin{aligned} \|\mathbf{y}\|_{C^1_b([s,\infty);\mathcal{V}^{-2}(\Omega))} &\leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)} + \big\|\theta^2BM^2B^*\Phi\big\|_{H^1(s,\infty;\mathcal{V}^{-2}(\Omega))} + \lambda_0\|\mathbf{y}\|_{H^1(s,\infty;\mathcal{V}^{-2}(\Omega))}\big) \\ &\leqslant C|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}. \end{aligned}$$

(Here $C^1_b([s,\infty);\mathcal{V}^{-2}(\Omega))$ denotes the space of functions belonging to $C_b([s,\infty);\mathcal{V}^{-2}(\Omega))$ whose first order time derivative belongs to $C_b([s,\infty);\mathcal{V}^{-2}(\Omega))$.)

From the equation satisfied by Φ, from estimate (5.4), and from the estimate of $\mathbf{y}$ in $H^1(s,\infty;\mathcal{V}^{-1}(\Omega))$, we deduce that Φ' belongs to $H^1(s,\infty;\mathbf{V}^0_n(\Omega))$. In particular $\Phi'(\infty) = 0$. Thus the function Φ also obeys the equation

$$\begin{aligned} &-\Phi'' = A^*\Phi' + (-A_0)^{-1}\mathbf{y}' = \big(A^* - \lambda_0 I\big)\Phi' + (-A_0)^{-1}\big(\mathbf{y}' + \lambda_0(-A_0)\Phi'\big) \quad \text{in } (s,\infty), \\ &\Phi'(\infty) = 0. \end{aligned}$$

Since Φ' belongs to $H^{1/2}(s,\infty;\mathcal{V}^2(\Omega))$ and $\mathbf{y}'$ belong to $L^2(s,\infty;\mathcal{V}^{-1}(\Omega))$, with estimate (5.4), we have

$$\big\|\Phi'\big\|_{C_b([s,\infty);\mathcal{V}^2(\Omega))} \leqslant C\big\|\mathbf{y}' + \lambda_0(-A_0)\Phi'\big\|_{L^2(s,\infty;\mathcal{V}^{-1}(\Omega))} \leqslant C|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}.$$

In particular, we have

$$\big|\Phi'(s)\big|_{\mathcal{V}^2(\Omega)} \leqslant C|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}.$$

Corollary 5.1. *If $\mathbf{y}_0 \in \mathbf{V}^{1/2-\varepsilon}_n(\Omega)$ for some $0 < \varepsilon \leqslant 1/2$, then the solution $(\mathbf{y}, \Phi)$ of system (5.3) belongs to $\mathbf{V}^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_{s,\infty}) \times (L^2(s,\infty;\mathbf{V}^{11/2-\varepsilon}(\Omega)) \cap H^{7/4-\varepsilon/2}(s,\infty;\mathbf{V}^2(\Omega)))$, and we have*:

$$
\begin{aligned}
&\|\mathbf{y}\|_{\mathbf{V}^{3/2-\varepsilon,3/4-\varepsilon/2}(Q_{s,\infty})} + \|\Phi\|_{L^2(s,\infty;\mathbf{V}^{11/2-\varepsilon}(\Omega))\cap H^{7/4-\varepsilon/2}(s,\infty;\mathbf{V}^2(\Omega))} \leqslant C|\mathbf{y}_0|_{\mathbf{V}_n^{1/2-\varepsilon}(\Omega)},\\
&\|B^*\Phi\|_{L^2(s,\infty;\mathbf{V}^{4-\varepsilon}(\Gamma))\cap H^{7/4-\varepsilon/2}(s,\infty;\mathbf{V}^{1/2}(\Gamma))} \leqslant C|\mathbf{y}_0|_{\mathbf{V}_n^{1/2-\varepsilon}(\Omega)},
\end{aligned}
\tag{5.5}
$$

where the constant C depends on $0<\varepsilon\leqslant 1/2$, but is independent of $s\in[0,\infty)$.

If $\mathbf{y}_0\in\mathbf{V}_0^{1/2+\varepsilon}(\Omega)$ for some $0<\varepsilon\leqslant 1/2$ and if $s=0$, then the solution $(\mathbf{y},\Phi)$ of system (5.3) *belongs to $\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)\times(L^2(0,\infty;\mathbf{V}^{11/2+\varepsilon}(\Omega))\cap H^{7/4+\varepsilon/2}(0,\infty;\mathcal{V}^2(\Omega)))$, and we have*:

$$
\|\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} + \|\Phi\|_{L^2(0,\infty;\mathbf{V}^{11/2+\varepsilon}(\Omega))\cap H^{7/4+\varepsilon/2}(0,\infty;\mathcal{V}^2(\Omega))} \leqslant C|\mathbf{y}_0|_{\mathbf{V}_0^{1/2+\varepsilon}(\Omega)},
$$

and

$$
\|B^*\Phi\|_{L^2(0,\infty;\mathbf{V}^{4+\varepsilon}(\Gamma))\cap H^{7/4+\varepsilon/2}(0,\infty;\mathbf{V}^{1/2}(\Gamma))} \leqslant C|\mathbf{y}_0|_{\mathbf{V}_0^{1/2+\varepsilon}(\Omega)}.
$$

Proof. *Step 1.* The estimate for $\mathbf{y}$, Φ, and $B^*\Phi$, when $\mathbf{y}_0\in\mathbf{V}_n^{1/2-\varepsilon}(\Omega)$ can be proved as in Corollary 4.1.

Step 2. Let us now suppose that $\mathbf{y}_0\in\mathbf{V}_0^{1/2+\varepsilon}(\Omega)$ for some $0<\varepsilon\leqslant 1/2$ and that $s=0$. From the first part we deduce that $B^*\Phi$ belongs to $L^2(0,\infty;\mathbf{V}^{7/2-\varepsilon'}(\Gamma))\cap H^{7/4-\varepsilon'/2}(0,\infty;\mathbf{V}^{1/2}(\Gamma))$ for all $\varepsilon'>0$. In particular $B^*\Phi$ belongs to $\mathbf{V}^{2,1}(\Sigma_\infty)$. Since $-\theta MB^*\Phi$ obeys the compatibility condition $-\theta MB^*\Phi|_{t=0}=0=\mathbf{y}_0|_\Gamma$, with Lemmas A.1 and A.3, we can deduce the estimate for $\mathbf{y}$ in $\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)$. The estimates for Φ and $B^*\Phi$ can next be obtained with Lemma A.5. □

Corollary 5.2. *The mapping $\Pi(\cdot)$ belongs to $C_s([0,2t_0];\mathcal{L}(\mathbf{V}_n^0(\Omega)))$. For all $0<\varepsilon\leqslant 1/2$, there exists a constant $C(\varepsilon)>0$, such that, for all $t\in[0,2t_0]$, we have*

$$
\|\Pi(t)\|_{\mathcal{L}(\mathbf{V}_n^{1/2-\varepsilon}(\Omega),\mathbf{V}^{9/2-\varepsilon}(\Omega)\cap\mathbf{V}_0^1(\Omega))} + \|B^*\Pi(t)\|_{\mathcal{L}(\mathbf{V}_n^{1/2-\varepsilon}(\Omega),\mathbf{V}^{3-\varepsilon}(\Gamma))} \leqslant C(\varepsilon).
$$

Proof. The fact that Π belongs to $C_s([0,2t_0];\mathcal{L}(\mathbf{V}_n^0(\Omega)))$ is a classical result [17, Theorem 1.2.2.1]. Since $\Pi(s)\mathbf{y}_0=\Phi^s_{\mathbf{y}_0}(s)$, where $(\mathbf{y}^s_{\mathbf{y}_0},\Phi^s_{\mathbf{y}_0})$ is the solution of system (5.3), the estimates for $\Pi(t)$ and $B^*\Pi(t)$ follows from estimate (5.5). □

Theorem 5.2. *The family of operators $(G(t,s))_{0\leqslant s\leqslant t}$ defined by*

$$
G(t,s)\mathbf{y}_0=\mathbf{y}^s_{\mathbf{y}_0}(t),
$$

where $(\mathbf{y}^s_{\mathbf{y}_0},\mathbf{u}^s_{\mathbf{y}_0})$ is the optimal solution to problem $(\mathcal{Q}_{s,\mathbf{y}_0})$, is a strongly continuous evolution operator on $\mathbf{V}_n^0(\Omega)$. Its infinitesimal generator is the family of unbounded operators $(A_\Pi(t),D(A_\Pi(t)))_{t\geqslant 0}$ defined by:

$$
\begin{aligned}
&D(A_\Pi(t))=\{\mathbf{y}\in\mathbf{V}_n^0(\Omega)\mid A\mathbf{y}-\theta^2(t)BM^2B^*\Pi(t)\mathbf{y}\in\mathbf{V}_n^0(\Omega)\},\\
&A_\Pi(t)\mathbf{y}=A\mathbf{y}-\theta^2(t)BM^2B^*\Pi(t)\mathbf{y}.
\end{aligned}
$$

Moreover, there exist $M_1>0$ and $\omega>0$ such that

$$
|G(t,s)\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)}\leqslant M_1\mathrm{e}^{-\omega(t-s)}|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)}\quad \textit{for all } t\geqslant s\geqslant 0.
\tag{5.6}
$$

Proof. The fact that the family of operators $(G(t,s))_{0\leqslant s\leqslant t}$ is a strongly continuous evolution operator on $\mathbf{V}_n^0(\Omega)$ follows from the dynamic programming principle and from the estimate stated in Lemma 5.1. Since $\mathbf{y}^s_{\mathbf{y}_0}$ is the solution of the equation

$$
\mathbf{y}'=A_\Pi(t)\mathbf{y}\quad \text{in } (s,\infty),\qquad \mathbf{y}(s)=\mathbf{y}_0,
$$

it is clear that the family of operators $(A_\Pi(t),D(A_\Pi(t)))_{t\geqslant 0}$ is the generator of $(G(t,s))_{0\leqslant s\leqslant t}$ (the characterization of $D(A_\Pi(t))$ can be done as in [19, Theorem 4.5]). Let us show the estimate (5.6). Consider successively the three cases $0\leqslant s\leqslant t\leqslant t_0$, $0\leqslant s\leqslant t_0\leqslant t$, and $0\leqslant t_0\leqslant s\leqslant t$. Due to Corollary 4.1 and to Datko's theorem, we know that the semigroup $(\mathrm{e}^{tA_\Pi})_{t\geqslant 0}$ is exponentially stable on $\mathbf{V}_n^0(\Omega)$. Thus we have:

$$
|\mathrm{e}^{tA_\Pi}\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)}\leqslant M\mathrm{e}^{-\omega t}|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)}
$$

for some $M \geqslant 1$ and some $\omega > 0$.

If $0 \leqslant s \leqslant t \leqslant t_0$, we have

$$\left|G(t,s)\mathbf{y}_0\right|_{\mathbf{V}_n^0(\Omega)} \leqslant K_1|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)} \leqslant K_1 \mathrm{e}^{-\omega(t-s)}\mathrm{e}^{\omega t_0}|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)},$$

where $K_1 \geqslant 1$ depends on t_0, but is independent of $0 \leqslant s \leqslant t_0$ and $0 \leqslant t \leqslant t_0$. If $0 \leqslant s \leqslant t_0 \leqslant t$, we have

$$\begin{aligned}\left|G(t,s)\mathbf{y}_0\right|_{\mathbf{V}_n^0(\Omega)} &= \left|\mathrm{e}^{A_\Pi(t-t_0)}G(t_0,s)\mathbf{y}_0\right|_{\mathbf{V}_n^0(\Omega)} \leqslant M\mathrm{e}^{-\omega(t-t_0)}K_1|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)}\\ &\leqslant M\mathrm{e}^{-\omega(s-t_0)}K_1\mathrm{e}^{-\omega(t-s)}|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)} \leqslant M\mathrm{e}^{\omega t_0}K_1\mathrm{e}^{-\omega(t-s)}|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)}.\end{aligned}$$

If $0 \leqslant t_0 \leqslant s \leqslant t$, we have

$$\left|G(t,s)\mathbf{y}_0\right|_{\mathbf{V}_n^0(\Omega)} = \left|\mathrm{e}^{A_\Pi(t-s)}\mathbf{y}_0\right|_{\mathbf{V}_n^0(\Omega)} \leqslant M\mathrm{e}^{-\omega(t-s)}|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)}.$$

Thus it is sufficient to take $M_1 = M\mathrm{e}^{\omega t_0}K_1$. $\square$

In a similar way, we have the following theorem.

Theorem 5.3. *The family of unbounded operators $(A_\Pi^*(t), D(A_\Pi^*(t)))_{t\geqslant 0}$ defined by*:

$$\begin{aligned}&D\big(A_\Pi^*(t)\big) = \big\{\mathbf{y} \in \mathbf{V}_n^0(\Omega) \;\big|\; A^*\mathbf{y} - \theta^2(t)\big(BM^2B^*\Pi(t)\big)^*\mathbf{y} \in \mathbf{V}_n^0(\Omega)\big\},\\ &A_\Pi^*(t)\mathbf{y} = A^*\mathbf{y} - \theta^2(t)\big(BM^2B^*\Pi(t)\big)^*\mathbf{y},\end{aligned}$$

is the infinitesimal generator of an evolution operator $(G^*(t,s))_{0\leqslant s\leqslant t}$. This evolution operator satisfies the exponential stability estimate

$$\left|G^*(t,s)\mathbf{y}_0\right|_{\mathbf{V}_n^0(\Omega)} \leqslant M_1\mathrm{e}^{-\omega(t-s)}|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)} \quad \textit{for all } t \geqslant s \geqslant 0. \tag{5.7}$$

Thanks to Theorem 4.2 and to estimates proved in Corollary 5.2, we can establish the following theorem.

Theorem 5.4. *The mapping $\Pi(\cdot) \in C_s([0,\infty); \mathcal{L}(\mathbf{V}_n^0(\Omega)))$ is the unique weak solution to the differential Riccati equation*

$$\begin{aligned}&\Pi^*(t) = \Pi(t) \in \mathcal{L}\big(\mathbf{V}_n^0(\Omega)\big) \quad \textit{and} \quad \Pi(t) \geqslant 0,\\ &\textit{for all } \mathbf{y} \in \mathbf{V}_n^0(\Omega), \quad \Pi(t)\mathbf{y} \in \mathbf{V}^2(\Omega) \cap \mathbf{V}_0^1(\Omega) \quad \textit{and} \quad \left|\Pi(t)\mathbf{y}\right|_{\mathbf{V}^2(\Omega)} \leqslant C|\mathbf{y}|_{\mathbf{V}_n^0(\Omega)},\\ &\textit{for } t \geqslant t_0, \quad \Pi(t) = \widehat{\Pi}, \quad \textit{where } \widehat{\Pi} \textit{ is the solution to Eq. (4.2)},\\ &\textit{for } t \leqslant t_0, \quad \Pi \textit{ is the solution to the differential equation}\\ &-\Pi'(t) = A^*\Pi + \Pi A - \theta^2(t)\Pi BM^2B^*\Pi + (-A_0)^{-1}, \qquad (5.8)\\ &\Pi(t_0) = \widehat{\Pi}. \qquad (5.9)\end{aligned}$$

5.2. Problem $(\mathcal{Q}_{s,\mathbf{y}_0})$ with initial condition in $\mathcal{V}^{-2}(\Omega)$

Theorem 5.5. *For all $\mathbf{y}_0 \in \mathcal{V}^{-2}(\Omega)$ and all $s \in [0,\infty)$, problem $(\mathcal{Q}_{s,\mathbf{y}_0})$ admits a unique solution $(\mathbf{y}_{\mathbf{y}_0}^s, \mathbf{u}_{\mathbf{y}_0}^s)$. There exists $\Pi(s) \in \mathcal{L}(\mathcal{V}^{-2}(\Omega), \mathcal{V}^2(\Omega))$ such that the optimal cost is given by*

$$I\big(s, \mathbf{y}_{\mathbf{y}_0}^s, \mathbf{u}_{\mathbf{y}_0}^s\big) = \frac{1}{2}\big\langle \Pi(s)\mathbf{y}_0, \mathbf{y}_0\big\rangle_{\mathcal{V}^2(\Omega),\mathcal{V}^{-2}(\Omega)}.$$

Proof. The proof is completely analogous to the one of Theorem 4.3. $\square$

Lemma 5.2. *For every $\mathbf{y}_0 \in \mathcal{V}^{-2}(\Omega)$, system (5.3) admits a unique solution in $L^2(s,\infty;\mathcal{V}^{-1}(\Omega)) \times \mathbf{V}^{2,1}(Q_{s,\infty})$. This solution belongs to $(L^2(s,\infty;\mathcal{V}^{-1}(\Omega)) \cap H^1(s,\infty;\mathcal{V}^{-3}(\Omega))) \times (L^2(s,\infty;\mathbf{V}^3(\Omega)) \cap H^1(s,\infty;\mathbf{V}_0^1(\Omega)))$ and it satisfies*:

$$\|\mathbf{y}\|_{L^2(s,\infty;\mathcal{V}^{-1}(\Omega))\cap H^1(s,\infty;\mathcal{V}^{-3}(\Omega))} + \|\Phi\|_{L^2(s,\infty;\mathbf{V}^3(\Omega))\cap H^1(s,\infty;\mathbf{V}_0^1(\Omega))} \leqslant C|\mathbf{y}_0|_{\mathcal{V}^{-2}(\Omega)},$$

where the constant C is independent of s. The pair $(\mathbf{y}, -\theta MB^*\Phi)$ is the solution of $(\mathcal{Q}_{s,\mathbf{y}_0})$.

Proof. The proof is completely analogous to the one of Lemma 4.1. □

Corollary 5.3. *There exists a constant $C > 0$, such that, for all $t \in [0, \infty)$, we have*

$$\|\Pi(t)\|_{\mathcal{L}(\mathcal{V}^{-2}(\Omega),\mathcal{V}^2(\Omega))} \leqslant C \quad \text{and} \quad \|B^*\Pi(t)\|_{\mathcal{L}(\mathcal{V}^{-2}(\Omega),\mathbf{V}^{1/2}(\Gamma))} \leqslant C.$$

In addition, for all $\mathbf{y}_0 \in \mathcal{V}^{-2}(\Omega)$, and all $\zeta \in \mathbf{V}^0_n(\Omega)$, the mapping

$$s \mapsto \big(\Pi(s)\mathbf{y}_0, \zeta\big)_{\mathbf{V}^0_n(\Omega)}$$

is continuous on $[0, 2t_0]$.

Proof. *Step 1.* For all $s \in [0, \infty)$, $\Pi(s)\mathbf{y}_0 = \Phi^s_{\mathbf{y}_0}(s)$, where $(\mathbf{y}^s_{\mathbf{y}_0}, \Phi^s_{\mathbf{y}_0})$ is the solution of system (5.3). Thus the estimates for $\Pi(\cdot)$ and $B^*\Pi(\cdot)$ directly follow from estimates stated for $\Phi^s_{\mathbf{y}_0}$.

Step 2. Let us show that $(\Pi(\cdot)\mathbf{y}_0, \zeta)_{\mathbf{V}^0_n(\Omega)}$ is continuous on $[0, 2t_0]$ for all $\mathbf{y}_0 \in \mathcal{V}^{-2}(\Omega)$, and all $\zeta \in \mathbf{V}^0_n(\Omega)$. Let s be given fixed in $(0, 2t_0]$, let h be positive and assume that $s - h > 0$. Let $(\mathbf{y}^s_{\mathbf{y}_0}, \Phi^s_{\mathbf{y}_0})$ (respectively $(\mathbf{y}^s_\zeta, \Phi^s_\zeta)$) be the solution to system (5.3) corresponding to the initial condition $\mathbf{y}(s) = \mathbf{y}_0 \in \mathcal{V}^{-2}(\Omega)$ (respectively $\mathbf{y}(s) = \zeta \in \mathbf{V}^0_n(\Omega)$), and let $(\mathbf{y}^{s-h}_{\mathbf{y}_0}, \Phi^{s-h}_{\mathbf{y}_0})$ (respectively $(\mathbf{y}^{s-h}_\zeta, \Phi^{s-h}_\zeta)$) be the solution to system (5.3) corresponding to the initial condition $\mathbf{y}(s-h) = \mathbf{y}_0$ (respectively $\mathbf{y}(s-h) = \zeta$). To prove the left side continuity at s, we want to show that

$$\lim_{h\to 0}\big(\Pi(s-h)\mathbf{y}_0, \zeta\big)_{\mathbf{V}^0_n(\Omega)} = \big(\Pi(s)\mathbf{y}_0, \zeta\big)_{\mathbf{V}^0_n(\Omega)}. \tag{5.10}$$

The right side continuity can be checked in the same way. From the dynamic programming principle it follows that

$$\begin{aligned}&\big(\mathbf{y}^{s-h}_\zeta, \Phi^{s-h}_\zeta\big) = \big(\mathbf{y}^s_{\mathbf{y}^{s-h}_\zeta(s)}, \Phi^s_{\mathbf{y}^{s-h}_\zeta(s)}\big) \quad \text{and}\\ &\big(\mathbf{y}^{s-h}_{\mathbf{y}_0}, \Phi^{s-h}_{\mathbf{y}_0}\big) = \big(\mathbf{y}^s_{\mathbf{y}^{s-h}_{\mathbf{y}_0}(s)}, \Phi^s_{\mathbf{y}^{s-h}_{\mathbf{y}_0}(s)}\big) \quad \text{on } [s, \infty).\end{aligned} \tag{5.11}$$

With estimates stated in Lemma 5.2 and Corollary 5.1 we have

$$\begin{aligned}&\big\|\mathbf{y}^s_{\mathbf{y}_0} - \mathbf{y}^s_{\mathbf{y}^{s-h}_{\mathbf{y}_0}(s)}\big\|_{L^2(s,\infty;\mathcal{V}^{-1}(\Omega))} + \big\|\Phi^s_{\mathbf{y}_0} - \Phi^s_{\mathbf{y}^{s-h}_{\mathbf{y}_0}(s)}\big\|_{\mathbf{V}^{2,1}(Q_{s,\infty})} \leqslant C\big|\mathbf{y}_0 - \mathbf{y}^{s-h}_{\mathbf{y}_0}(s)\big|_{\mathcal{V}^{-2}(\Omega)}, \quad \text{and}\\ &\big\|\mathbf{y}^s_\zeta - \mathbf{y}^s_{\mathbf{y}^{s-h}_\zeta(s)}\big\|_{L^2(s,\infty;\mathbf{V}^0_n(\Omega))} + \big\|\Phi^s_\zeta - \Phi^s_{\mathbf{y}^{s-h}_\zeta(s)}\big\|_{\mathbf{V}^{2,1}(Q_{s,\infty})} \leqslant C\big|\zeta - \mathbf{y}^{s-h}_\zeta(s)\big|_{\mathbf{V}^0_n(\Omega)}.\end{aligned} \tag{5.12}$$

We can show that

$$\begin{aligned}\big(\Pi(s-h)\mathbf{y}_0, \zeta\big)_{\mathbf{V}^0_n(\Omega)} = {}&\int_{s-h}^{\infty} \big((-A_0)^{-1}\mathbf{y}^{s-h}_{\mathbf{y}_0}(\tau), \mathbf{y}^{s-h}_\zeta(\tau)\big)_{\mathbf{V}^0_n(\Omega)}\,\mathrm{d}\tau\\ &+ \int_{s-h}^{\infty} \big(\theta(\tau)MB^*\Phi^{s-h}_{\mathbf{y}_0}(\tau), \theta(\tau)MB^*\Phi^{s-h}_\zeta(\tau)\big)_{\mathbf{V}^0(\Gamma)}\,\mathrm{d}\tau,\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}\big(\Pi(s)\mathbf{y}_0, \zeta\big)_{\mathbf{V}^0_n(\Omega)} = {}&\int_{s}^{\infty} \big((-A_0)^{-1}\mathbf{y}^{s}_{\mathbf{y}_0}(\tau), \mathbf{y}^{s}_\zeta(\tau)\big)_{\mathbf{V}^0_n(\Omega)}\,\mathrm{d}\tau\\ &+ \int_{s}^{\infty} \big(\theta(\tau)MB^*\Phi^{s}_{\mathbf{y}_0}(\tau), \theta(\tau)MB^*\Phi^{s}_\zeta(\tau)\big)_{\mathbf{V}^0(\Gamma)}\,\mathrm{d}\tau.\end{aligned} \tag{5.14}$$

Using the estimates in Lemmas 5.1 and 5.2, we can show that

$$\lim_{h\to 0}\left(\int_{s-h}^{s}\big((-A_0)^{-1}\mathbf{y}_{\mathbf{y}_0}^{s-h}(\tau),\mathbf{y}_{\zeta}^{s-h}(\tau)\big)_{\mathbf{V}_n^0(\Omega)}\,\mathrm{d}\tau + \int_{s-h}^{s}\big(\theta(\tau)MB^*\Phi_{\mathbf{y}_0}^{s-h}(\tau),\theta(\tau)MB^*\Phi_{\zeta}^{s-h}(\tau)\big)_{\mathbf{V}^0(\Gamma)}\,\mathrm{d}\tau\right)=0.$$

Now (5.10) may be deduced from (5.11)–(5.14). □

5.3. Regularity of Π'

Let us denote by $(\mathbf{y}_{\mathbf{y}_0}^s, \Phi_{\mathbf{y}_0}^s)$ the solution to system (5.3). Due to Remark 5.1, if $\mathbf{y}_0 \in \mathbf{V}_n^0(\Omega)$, then $(\mathbf{y}_{\mathbf{y}_0}^s)'(s)$ is well defined in $\mathcal{V}^{-2}(\Omega)$ by

$$\big(\mathbf{y}_{\mathbf{y}_0}^s\big)'(s) = A\mathbf{y}_0 - \theta^2 BM^2B^*\Phi_{\mathbf{y}_0}^s(s).$$

Theorem 5.6. *Let Π be the solution to system* (5.8)–(5.9). *For all $\mathbf{y}_0 \in \mathbf{V}_n^0(\Omega)$, and $\zeta \in \mathbf{V}_n^0(\Omega)$, the mapping*

$$s \mapsto \big(\Pi(s)\mathbf{y}_0, \zeta\big)_{\mathbf{V}_n^0(\Omega)}$$

is differentiable on $[0, 2t_0]$. *Its derivative obeys*

$$\big(\Pi'(s)\mathbf{y}_0, \zeta\big)_{\mathbf{V}_n^0(\Omega)} = \big(\big(\Phi_{\mathbf{y}_0}^s\big)'(s) - \Pi(s)\big(\mathbf{y}_{\mathbf{y}_0}^s\big)'(s), \zeta\big)_{\mathbf{V}_n^0(\Omega)} \quad \textit{for all } s \in [0, 2t_0],$$

where $(\mathbf{y}_{\mathbf{y}_0}^s, \Phi_{\mathbf{y}_0}^s)$ is the solution to system (5.3)*, and we have*

$$\big|\Pi'(s)\mathbf{y}_0\big|_{\mathcal{V}^2(\Omega)} \leqslant C|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)} \quad \textit{for all } s \in [0, 2t_0].$$

Remark 5.2. From Theorem 5.4, it follows that

$$\begin{aligned}\big(\Pi'(s)\mathbf{y}_0, \zeta\big)_{\mathbf{V}_n^0(\Omega)} &= -\big(A^*\Pi(s)\mathbf{y}_0 + \Pi(s)A\mathbf{y}_0 - \theta^2(s)\Pi(s)BM^2B^*\Pi(s)\mathbf{y}_0 + (-A_0)^{-1}\mathbf{y}_0, \zeta\big)_{\mathbf{V}_n^0(\Omega)}\\ &= \big(-A^*\Phi_{\mathbf{y}_0}^s(s) - (-A_0)^{-1}\mathbf{y}_0 - \Pi(s)\big(A\mathbf{y}_0 - \theta^2BM^2B^*\Phi_{\mathbf{y}_0}^s(s)\big), \zeta\big)_{\mathbf{V}_n^0(\Omega)}\\ &= \big(\big(\Phi_{\mathbf{y}_0}^s\big)'(s) - \Pi(s)\big(\mathbf{y}_{\mathbf{y}_0}^s\big)'(s), \zeta\big)_{\mathbf{V}_n^0(\Omega)},\end{aligned}$$

which provides a short proof of the identity stated in the theorem. However, we have not given a complete proof of Theorem 5.4, because Theorem 5.4 is not essential in our analysis. It is only stated to highlight the fact that the family of operators $(\Pi(t))_{t\geqslant 0}$ obeys a well posed Riccati equation. Here, we give a proof of Theorem 5.6 independent of the result stated in Theorem 5.4.

Proof. *Step 1.* We first show that the mapping

$$s \mapsto \big(\Pi(s)\mathbf{y}_0, \zeta\big)_{\mathbf{V}_n^0(\Omega)}$$

admits a right side derivative equal to $((\Phi_{\mathbf{y}_0}^s)'(s) - \Pi(s)(\mathbf{y}_{\mathbf{y}_0}^s)'(s), \zeta)_{\mathbf{V}_n^0(\Omega)}$. Let s be in $[0, 2t_0]$, $\mathbf{y}_0 \in \mathbf{V}_n^0(\Omega)$, $\zeta \in \mathbf{V}_n^0(\Omega)$, and h be positive. From the definition of $\Pi(s)$, it follows that

$$\Pi(s+h)\mathbf{y}_0 - \Pi(s)\mathbf{y}_0 = \Phi_{\mathbf{y}_0}^{s+h}(s+h) - \Phi_{\mathbf{y}_0}^s(s+h) + \Phi_{\mathbf{y}_0}^s(s+h) - \Phi_{\mathbf{y}_0}^s(s),$$

where $(\mathbf{y}_{\mathbf{y}_0}^{s+h}, \Phi_{\mathbf{y}_0}^{s+h})$ is the solution to system (5.3) corresponding to the initial condition $\mathbf{y}(s+h) = \mathbf{y}_0$. Since $\Phi_{\mathbf{y}_0}^s$ belongs to $C^1([0,\infty); \mathcal{V}^2(\Omega))$, we have

$$\lim_{h\searrow 0}\frac{1}{h}\big(\Phi_{\mathbf{y}_0}^s(s+h) - \Phi_{\mathbf{y}_0}^s(s), \zeta\big)_{\mathbf{V}_n^0(\Omega)} = \big(\big(\Phi_{\mathbf{y}_0}^s\big)'(s), \zeta\big)_{\mathbf{V}_n^0(\Omega)}.$$

For the other term, we have

$$\frac{1}{h}\big(\Phi^{s+h}_{\mathbf{y}_0}(s+h)-\Phi^{s}_{\mathbf{y}_0}(s+h)\big)=\frac{1}{h}\big(\Phi^{s+h}_{\mathbf{y}_0}(s+h)-\Phi^{s+h}_{\mathbf{y}^s_{\mathbf{y}_0}(s+h)}(s+h)\big)$$
$$=\Pi(s+h)\bigg(\frac{\mathbf{y}_0-\mathbf{y}^s_{\mathbf{y}_0}(s+h)}{h}\bigg).$$

Since $\mathbf{y}^s_{\mathbf{y}_0}$ belongs to $C^1([s,\infty);\mathcal{V}^{-2}(\Omega))$, for all $\varepsilon>0$, there exists a function $h(\varepsilon)>0$, such that

$$\left|\frac{1}{h}\big(\mathbf{y}_0-\mathbf{y}^s_{\mathbf{y}_0}(s+h)\big)+\big(\mathbf{y}^s_{\mathbf{y}_0}\big)'(s)\right|_{\mathcal{V}^{-2}(\Omega)}\leqslant\varepsilon\quad\text{for all } h\in\big(0,h(\varepsilon)\big).$$

We can write

$$\begin{aligned}&\big(\Pi(s+h)\big(\mathbf{y}_0-\mathbf{y}^s_{\mathbf{y}_0}(s+h)\big)/h,\zeta\big)_{\mathbf{V}^0_n(\Omega)}\\&\quad=\big(\Pi(s)\big(\mathbf{y}_0-\mathbf{y}^s_{\mathbf{y}_0}(s+h)\big)/h,\zeta\big)_{\mathbf{V}^0_n(\Omega)}-\big(\big(\Pi(s+h)-\Pi(s)\big)\big(\mathbf{y}^s_{\mathbf{y}_0}\big)'(s),\zeta\big)_{\mathbf{V}^0_n(\Omega)}\\&\qquad+\big(\big(\Pi(s+h)-\Pi(s)\big)\big[\big(\mathbf{y}_0-\mathbf{y}^s_{\mathbf{y}_0}(s+h)\big)/h+\big(\mathbf{y}^s_{\mathbf{y}_0}\big)'(s)\big],\zeta\big)_{\mathbf{V}^0_n(\Omega)}.\end{aligned}$$

With Corollary 5.3, we have

$$\lim_{h\searrow 0}\big(\big(\Pi(s+h)-\Pi(s)\big)\big(\mathbf{y}^s_{\mathbf{y}_0}\big)'(s),\zeta\big)_{\mathbf{V}^0_n(\Omega)}=0.$$

Thus, with the estimate

$$\big|\big(\big(\Pi(s+h)-\Pi(s)\big)\big[\big(\mathbf{y}_0-\mathbf{y}^s_{\mathbf{y}_0}(s+h)\big)/h+\big(\mathbf{y}^s_{\mathbf{y}_0}\big)'(s)\big],\zeta\big)_{\mathbf{V}^0_n(\Omega)}\big|\leqslant C\varepsilon,$$

we deduce that

$$\lim_{h\searrow 0}\big(\Pi(s+h)\big(\mathbf{y}_0-\mathbf{y}^s_{\mathbf{y}_0}(s+h)\big)/h,\zeta\big)_{\mathbf{V}^0_n(\Omega)}=-\big(\Pi(s)\big(\mathbf{y}^s_{\mathbf{y}_0}\big)'(s),\zeta\big)_{\mathbf{V}^0_n(\Omega)}.$$

Therefore, we have shown that

$$\lim_{h\searrow 0}\frac{1}{h}\big(\big(\Pi(s+h)-\Pi(s)\big)\mathbf{y}_0,\zeta\big)_{\mathbf{V}^0_n(\Omega)}=\big(\big(\Phi^s_{\mathbf{y}_0}\big)'(s)-\Pi(s)\big(\mathbf{y}^s_{\mathbf{y}_0}\big)'(s),\zeta\big)_{\mathbf{V}^0_n(\Omega)},\tag{5.15}$$

for all $s\in[0,2t_0]$.

Step 2. We have (see also Remark 5.2)

$$\begin{aligned}&\big(\big(\Phi^s_{\mathbf{y}_0}\big)'(s)-\Pi(s)\big(\mathbf{y}^s_{\mathbf{y}_0}\big)'(s),\zeta\big)_{\mathbf{V}^0_n(\Omega)}\\&\quad=\big(-A^*\Phi^s_{\mathbf{y}_0}(s)-(-A_0)^{-1}\mathbf{y}_0-\Pi(s)\big(A\mathbf{y}_0-\theta^2BM^2B^*\Phi^s_{\mathbf{y}_0}(s)\big),\zeta\big)_{\mathbf{V}^0_n(\Omega)}\\&\quad=\big(-A^*\Pi(s)\mathbf{y}_0-(-A_0)^{-1}\mathbf{y}_0-\Pi(s)\big(A\mathbf{y}_0-\theta^2BM^2B^*\Pi(s)\mathbf{y}_0(s)\big),\zeta\big)_{\mathbf{V}^0_n(\Omega)}.\end{aligned}$$

From this identity, we can easily deduce that the mapping

$$s\mapsto\big(\big(\Phi^s_{\mathbf{y}_0}\big)'(s)-\Pi(s)\big(\mathbf{y}^s_{\mathbf{y}_0}\big)'(s),\zeta\big)_{\mathbf{V}^0_n(\Omega)}$$

is continuous on $[0,2t_0]$. Since the mapping

$$s\mapsto\big(\Pi(s)\mathbf{y}_0,\zeta\big)_{\mathbf{V}^0_n(\Omega)}$$

is continuous on $[0,2t_0]$, and its right-hand side derivative is also continuous on $[0,2t_0]$, we deduce that it is of class C^1 on $[0,2t_0]$, and that its derivative is identical to its right-hand side derivative. The first part of the theorem is proved.

Step 3. To prove the estimate for $\Pi'(s)$, we have to notice that, due to Remark 5.1, we have

$$\big|\big(\Phi^s_{\mathbf{y}_0}\big)'(s)\big|_{\mathcal{V}^2(\Omega)}+\big|\big(\mathbf{y}^s_{\mathbf{y}_0}\big)'(s)\big|_{\mathcal{V}^{-2}(\Omega)}\leqslant C|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}.$$

Thus

$$\big|\Pi'(s)\mathbf{y}_0\big|_{\mathcal{V}^2(\Omega)}\leqslant C|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)},$$

and the proof is complete. □

Corollary 5.4. *There exists a constant $C > 0$ such that, for all $t \in [0, 2t_0]$, $\tau \in [0, 2t_0]$, and all $\mathbf{y}_0 \in \mathbf{V}^0_n(\Omega)$,*

$$\big|\big(B^*\Pi(t) - B^*\Pi(\tau)\big)\mathbf{y}_0\big|_{\mathbf{V}^{1/2}(\Gamma)} \leqslant C|t-\tau|\,|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}.$$

Proof. For all $t \in [0, 2t_0]$, $\tau \in [0, 2t_0]$, $\mathbf{y}_0 \in \mathbf{V}^0_n(\Omega)$, and $\zeta \in \mathcal{V}^{-2}(\Omega)$, we have

$$\big\langle\big(\Pi(t) - \Pi(\tau)\big)\mathbf{y}_0, \zeta\big\rangle_{\mathcal{V}^2(\Omega),\mathcal{V}^{-2}(\Omega)} = (t-\tau)\int_0^1 \big\langle \Pi'\big(\tau + \theta(t-\tau)\big)\mathbf{y}_0, \zeta\big\rangle_{\mathcal{V}^2(\Omega),\mathcal{V}^{-2}(\Omega)}\,\mathrm{d}\theta.$$

Thus, with Theorem 5.6, we can write

$$\big|\big\langle\big(\Pi(t) - \Pi(\tau)\big)\mathbf{y}_0, \zeta\big\rangle_{\mathcal{V}^2(\Omega),\mathcal{V}^{-2}(\Omega)}\big| \leqslant C|t-\tau|\,|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}|\zeta|_{\mathcal{V}^{-2}(\Omega)},$$

from which we deduce

$$\big|\big(\Pi(t) - \Pi(\tau)\big)\mathbf{y}_0\big|_{\mathcal{V}^2(\Omega)} \leqslant C|t-\tau|\,|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}.$$

With Proposition 2.1, we finally obtain

$$\big|\big(B^*\Pi(t) - B^*\Pi(\tau)\big)\mathbf{y}_0\big|_{\mathbf{V}^{1/2}(\Gamma)} \leqslant C\big|\big(\Pi(t) - \Pi(\tau)\big)\mathbf{y}_0\big|_{\mathcal{V}^2(\Omega)} \leqslant C|t-\tau|\,|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}. \qquad \square$$

6. Problems with a nonhomogeneous source term

In this section, we want to study the regularity of the solution to the equation

$$\mathbf{y}' = A_\Pi(t)\mathbf{y} + \mathbf{f}, \quad \mathbf{y}(0) = \mathbf{y}_0, \tag{6.1}$$

when $\mathbf{y}_0 \in \mathbf{V}^{1/2+\varepsilon}_0(\Omega)$, and $\mathbf{f} \in L^1(0,\infty; \mathbf{V}^0_n(\Omega)) \cap L^2(0,\infty; (\mathbf{V}^{1/2-\varepsilon}_n(\Omega))')$, with $0 < \varepsilon \leqslant 1/2$. For that, we follow the method introduced in [19] in the two-dimensional case. We decompose the solution $\mathbf{y}$ to Eq. (6.1) in the form $\mathbf{y} = \hat{\mathbf{y}} + \tilde{\mathbf{y}}$, where $\hat{\mathbf{y}}$ is the optimal state of a control problem with a nonhomogeneous state equation, and $\tilde{\mathbf{y}}$ obeys an auxiliary equation. The regularity of $\tilde{\mathbf{y}}$ is studied in Lemma 6.2 by a bootstrap argument.

For all $\mathbf{y}_0 \in \mathbf{V}^0_n(\Omega)$, and $\mathbf{f} \in L^1(0,\infty; \mathbf{V}^0_n(\Omega))$, we consider the problem

$$\inf\big\{I(\mathbf{y},\mathbf{u}) \bigm| (\mathbf{y},\mathbf{u}) \text{ satisfies (6.2)},\ \mathbf{u} \in \mathbf{V}^{0,0}(\Sigma_\infty)\big\}, \tag{$\mathcal{Q}_{\mathbf{y}_0,\mathbf{f}}$}$$

where

$$I(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty\int_\Omega \big|(-A_0)^{-1/2}\mathbf{y}\big|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^\infty \big|\mathbf{u}(t)\big|^2_{\mathbf{V}^0(\Gamma)}\,\mathrm{d}t,$$

and

$$\mathbf{y}' = A\mathbf{y} + B\theta M\mathbf{u} + \mathbf{f} \quad \text{in } (0,\infty), \quad \mathbf{y}(0) = \mathbf{y}_0. \tag{6.2}$$

In this section we want to study the regularity of solutions to the control problem $(\mathcal{Q}_{\mathbf{y}_0,\mathbf{f}})$ in function of the regularity of $\mathbf{y}_0$, when $\mathbf{f}$ belongs to $L^1(0,\infty; \mathbf{V}^0_n(\Omega)) \cap L^2(0,\infty; (\mathbf{V}^{1/2-\varepsilon}(\Omega))')$, with $0 < \varepsilon \leqslant 1/2$. This result will be used in the next section to study the local stabilization of the three-dimensional Navier–Stokes equations.

Theorem 6.1. *For all $\mathbf{y}_0 \in \mathbf{V}^0_n(\Omega)$, $\mathbf{f} \in L^1(0,\infty; \mathbf{V}^0_n(\Omega))$, problem $(\mathcal{Q}_{\mathbf{y}_0,\mathbf{f}})$ admits a unique solution $(\mathbf{y}_{\mathbf{y}_0,\mathbf{f}}, \mathbf{u}_{\mathbf{y}_0,\mathbf{f}})$, and the optimal cost obeys*

$$I(\mathbf{y}_{\mathbf{y}_0,\mathbf{f}}, \mathbf{u}_{\mathbf{y}_0,\mathbf{f}}) \leqslant C\big(|\mathbf{y}_0|^2_{\mathbf{V}^0_n(\Omega)} + \|\mathbf{f}\|^2_{L^1(0,\infty;\mathbf{V}^0_n(\Omega))}\big). \tag{6.3}$$

Proof. Since the evolution operator $(G(t,s))_{0\leqslant s\leqslant t}$ satisfies the exponential stability condition (5.6), the solution to the equation

$$\mathbf{z}' = A_\Pi(t)\mathbf{z} + \mathbf{f} \quad \text{in } (0,\infty), \quad \mathbf{z}(0) = \mathbf{y}_0,$$

defined by

$$\mathbf{z}(t) = G(t,0)\mathbf{y}_0 + \int_0^t G(t,\tau)\mathbf{f}(\tau)\,\mathrm{d}\tau,$$

obeys the estimate

$$\big|\mathbf{z}(t)\big|_{\mathbf{V}_n^0(\Omega)} \leqslant C\mathrm{e}^{-\omega t}|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)} + C\int_0^t \mathrm{e}^{-\omega(t-\tau)}\big|\mathbf{f}(\tau)\big|_{\mathbf{V}_n^0(\Omega)}\,\mathrm{d}\tau,$$

for some $C > 0$. It follows that

$$\|\mathbf{z}\|_{L^2(0,\infty;\mathbf{V}_n^0(\Omega))} \leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))}\big).$$

Since

$$\big\|\theta(t)MB^*\Pi(t)\big\|_{\mathcal{L}(\mathbf{V}_n^0(\Omega),\mathbf{V}^0(\Gamma))} \leqslant C \quad \text{for all } t \geqslant 0$$

(see Corollary 5.2), we also have

$$\|\theta MB^*\Pi\mathbf{z}\|_{L^2(0,\infty;\mathbf{V}^0(\Gamma))} \leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))}\big).$$

The pair $(\mathbf{z}, -\theta MB^*\Pi\mathbf{z})$ is admissible for $(\mathcal{Q}_{\mathbf{y}_0,\mathbf{f}})$ and we have

$$I(\mathbf{z}, -\theta MB^*\Pi\mathbf{z}) \leqslant C\big(|\mathbf{y}_0|^2_{\mathbf{V}_n^0(\Omega)} + \|\mathbf{f}\|^2_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))}\big).$$

Therefore by classical arguments we can prove that $(\mathcal{Q}_{\mathbf{y}_0,\mathbf{f}})$ admits a unique solution $(\mathbf{y}_{\mathbf{y}_0,\mathbf{f}}, \mathbf{u}_{\mathbf{y}_0,\mathbf{f}})$ (see [17, proof of Theorem 2.3.3.1 (i), p. 135]). Estimate (6.3) follows from the above inequality satisfied by $I(\mathbf{z}, -\theta MB^*\Pi\mathbf{z})$. □

Lemma 6.1. *For all* $\mathbf{u} \in L^2(0,\infty;\mathbf{V}^0(\Gamma))$, $\mathbf{y} \in L^2(0,\infty;\mathbf{V}_n^0(\Omega))$, *the equation*

$$-\Phi' = A_\Pi^*(t)\Phi - \big(\theta(t)MB^*\Pi(t)\big)^*\mathbf{u} + (-A_0)^{-1}\mathbf{y} \quad \text{in } (0,\infty), \qquad \Phi(\infty) = 0, \tag{6.4}$$

admits a unique solution in $L^2(0,\infty;\mathbf{V}_n^0(\Omega))$ *and*

$$\|\Phi\|_{L^2(0,\infty;\mathbf{V}_n^0(\Omega))} + \|\Phi\|_{L^\infty(0,\infty;\mathbf{V}_n^0(\Omega))} \leqslant C\big(\big\|(-A_0)^{-1}\mathbf{y}\big\|_{L^2(0,\infty;\mathbf{V}_n^0(\Omega))} + \|\mathbf{u}\|_{L^2(0,\infty;\mathbf{V}^0(\Gamma))}\big).$$

Proof. We already know that

$$\big\|\big(\theta(t)MB^*\Pi(t)\big)^*\big\|_{\mathcal{L}(\mathbf{V}^0(\Gamma),\mathbf{V}_n^0(\Omega))} \leqslant C \quad \text{for all } t \geqslant 0.$$

Thus from the exponential stability of the evolution operator $(G^*(t,s))_{0\leqslant s\leqslant t}$ it yields

$$\big|\Phi(t)\big|_{\mathbf{V}_n^0(\Omega)} \leqslant C\int_t^\infty \mathrm{e}^{-\omega(\tau-t)}\big(\big|(-A_0)^{-1}\mathbf{y}(\tau)\big|_{\mathbf{V}_n^0(\Omega)} + \big|\mathbf{u}(\tau)\big|_{\mathbf{V}^0(\Gamma)}\big)\,\mathrm{d}\tau.$$

The estimates of the lemma follows from Young's inequality for convolutions. The uniqueness of solution is obvious. □

Lemma 6.2. *Let* $\mathbf{f}$ *be in* $L^1(0,\infty;\mathbf{V}_n^0(\Omega))$, $\mathbf{y}_0$ *be in* $\mathbf{V}_n^0(\Omega)$, *denote by* $(\hat{\mathbf{y}}, \hat{\mathbf{u}})$ *the solution to problem* $(\mathcal{Q}_{0,\mathbf{y}_0,\mathbf{f}})$, *and let* $\hat{\Phi}$ *be the solution to Eq.* (6.4) *corresponding to* $(\hat{\mathbf{y}}, \hat{\mathbf{u}})$. *Then* $(\hat{\mathbf{y}}, \hat{\Phi})$ *is also solution in* $L^2(0,\infty;\mathbf{V}_n^0(\Omega)) \times L^2(0,\infty;\mathbf{V}_n^0(\Omega))$ *to the system*

$$\begin{aligned} &\mathbf{y}' = A\mathbf{y} - B\theta^2M^2B^*\Phi + \mathbf{f} \quad \text{in } (0,\infty), \qquad \mathbf{y}(0) = \mathbf{y}_0, \\ &-\Phi' = A^*\Phi + (-A_0)^{-1}\mathbf{y} \quad \text{in } (0,\infty), \qquad \Phi(\infty) = 0. \end{aligned} \tag{6.5}$$

The following estimate holds

$$\|\hat{\mathbf{y}}\|_{L^2(0,\infty;\mathbf{V}_n^0(\Omega))} + \|\hat{\Phi}\|_{L^2(0,\infty;\mathbf{V}^4(\Omega))} + \|\hat{\Phi}\|_{H^1(0,\infty;\mathbf{V}^2(\Omega))} \leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))}\big).$$

Proof. *Step 1.* Due to estimate (6.3), we have

$$\|(-A_0)^{-1/2}\hat{\mathbf{y}}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))} + \|\hat{\mathbf{u}}\|_{L^2(0,\infty;\mathbf{V}^0(\Gamma))} \leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}^0_n(\Omega))}\big).$$

Consider the problem

$$\inf\big\{I_k(0,\mathbf{y},\mathbf{u}) \;\big|\; (\mathbf{y},\mathbf{u}) \text{ satisfies (6.6)},\ \mathbf{u}\in\mathbf{V}^{0,0}(\Sigma_{0,k})\big\}, \tag{$\mathcal{Q}^k_{0,\mathbf{y}_0,\mathbf{f}}$}$$

where

$$I_k(0,\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^k\int_\Omega \big|(-A_0)^{-1/2}\mathbf{y}\big|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^k \big|\mathbf{u}(t)\big|^2_{\mathbf{V}^0(\Gamma)}\,\mathrm{d}t,$$

and

$$\mathbf{y}' = A\mathbf{y} + \theta BM\mathbf{u} + \mathbf{f} \quad \text{in } (0,k), \qquad \mathbf{y}(0) = \mathbf{y}_0. \tag{6.6}$$

Problem ($\mathcal{Q}^k_{0,\mathbf{y}_0,\mathbf{f}}$) admits a unique solution $(\mathbf{y}_k,\mathbf{u}_k)$ characterized by

$$\begin{aligned}
&\mathbf{y}_k' = A\mathbf{y}_k + \theta BM\mathbf{u}_k + \mathbf{f} \quad \text{in } (0,k), \qquad \mathbf{y}_k(0) = \mathbf{y}_0,\\
&-\Phi_k' = A^*\Phi_k + (-A_0)^{-1}\mathbf{y}_k \quad \text{in } (0,k), \qquad \Phi_k(k) = 0,\\
&\mathbf{u}_k = -\theta MB^*\Phi_k.
\end{aligned} \tag{6.7}$$

Since we have

$$\begin{aligned}
&\int_0^k\int_\Omega \big|(-A_0)^{-1/2}\mathbf{y}_k\big|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^k \big|\mathbf{u}_k(t)\big|^2_{\mathbf{V}^0(\Gamma)}\,\mathrm{d}t\\
&\quad\leqslant \int_0^\infty\int_\Omega \big|(-A_0)^{-1/2}\hat{\mathbf{y}}\big|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \big|\hat{\mathbf{u}}(t)\big|^2_{\mathbf{V}^0(\Gamma)}\,\mathrm{d}t,
\end{aligned}$$

as in the proof of Theorem 3.1, we can show that

$$\tilde{\mathbf{u}}_k \to \hat{\mathbf{u}} \quad \text{in } L^2\big(0,\infty;\mathbf{V}^0(\Gamma)\big) \quad \text{and} \quad \tilde{\mathbf{y}}_k \to \hat{\mathbf{y}} \text{ in } L^2\big(0,\infty;\mathcal{V}^{-1}(\Omega)\big), \tag{6.8}$$

where $\tilde{\mathbf{u}}_k$ and $\tilde{\mathbf{y}}_k$ denote the extensions by zero of $\mathbf{u}_k$ and $\mathbf{y}_k$ to (k,∞).

Step 2. Passage to the limit for Φ_k. Let $\tilde{\Phi}_k$ be the extension by zero of Φ_k to (k,∞). We have

$$-\tilde{\Phi}_k' = A^*\tilde{\Phi}_k + (-A_0)^{-1}\tilde{\mathbf{y}}_k \quad \text{in } (0,\infty), \qquad \tilde{\Phi}_k(\infty) = 0.$$

We already know that

$$\big\|\big(\theta(t)MB^*\Pi(t)\big)^*\big\|_{\mathcal{L}(\mathbf{V}^0(\Gamma),\mathbf{V}^0_n(\Omega))} \leqslant C \quad \text{for all } t \geqslant 0.$$

We can rewrite the above equation in the form

$$-\tilde{\Phi}_k' = A^*_\Pi(t)\tilde{\Phi}_k - \big(\theta(t)MB^*\Pi(t)\big)^*\tilde{\mathbf{u}}_k + (-A_0)^{-1}\tilde{\mathbf{y}}_k, \qquad \tilde{\Phi}_k(\infty) = 0. \tag{6.9}$$

Due to (6.8) and to Lemma 6.1, we can claim that

$$\big\|\tilde{\Phi}_k - \hat{\Phi}\big\|_{L^\infty(0,\infty;\mathbf{V}^0_n(\Omega))} + \big\|\tilde{\Phi}_k - \hat{\Phi}\big\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))} \to 0 \quad \text{when } k\to\infty,$$

where $\hat{\Phi}$ is the solution of Eq. (6.4) corresponding to $(\hat{\mathbf{y}},\hat{\mathbf{u}})$. Notice that

$$\|\hat{\Phi}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))} \leqslant C\big(\|\hat{\mathbf{u}}\|_{L^2(0,\infty;\mathbf{V}^0(\Gamma))} + \|\hat{\mathbf{y}}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}\big) \leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}^0_n(\Omega))}\big). \tag{6.10}$$

By passing to the limit, when k tends to infinity, in the equation

$$\tilde{\Phi}_k(t) = \int_t^\infty \mathrm{e}^{(A^*-\lambda_0 I)(\tau-t)}\big((-A_0)^{-1}\tilde{\mathbf{y}}_k(\tau) + \lambda_0\tilde{\Phi}_k(\tau)\big)\,\mathrm{d}\tau,$$

with Lemmas A.5 and A.7, we can show that $(\tilde{\Phi}_k)_k$ converges to $\hat{\Phi}$ in $L^2(0,\infty;\mathbf{V}^4(\Omega))\cap H^1(0,\infty;\mathbf{V}^2(\Omega))$, $(B^*\tilde{\Phi}_k)_k$ converges to $B^*\hat{\Phi}$ in $L^2(0,\infty;\mathbf{V}^0(\Gamma))$, and $\hat{\Phi}$ satisfies

$$\hat{\Phi}(t)=\int_t^\infty \mathrm{e}^{(A^*-\lambda_0 I)(\tau-t)}\big((-A_0)^{-1}\hat{\mathbf{y}}(\tau)+\lambda_0\hat{\Phi}(\tau)\big)\,\mathrm{d}\tau.$$

Thus $\hat{\Phi}$ satisfies the second equation in (6.5) corresponding to $\hat{\mathbf{y}}$. Since $(\tilde{\mathbf{u}}_k)_k=(-\theta MB^*\tilde{\Phi}_k)_k$ converges to $\hat{\mathbf{u}}$, we have $\hat{\mathbf{u}}=-\theta MB^*\hat{\Phi}$, and $(\hat{\mathbf{y}},\hat{\Phi})$ obeys the system (6.5). The estimate for $\hat{\Phi}$ follows from Lemma A.5 and from (6.10). □

Theorem 6.2. *Assume that* $\mathbf{f}\in L^1(0,\infty;\mathbf{V}^0_n(\Omega))\cap L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}(\Omega))')$ *for some* $0<\varepsilon\leqslant 1/2$. *If* $\mathbf{y}_0\in\mathbf{V}^0_n(\Omega)$, *then the pair* $(\hat{\mathbf{y}},\hat{\Phi})$, *which obeys systems* (6.4) *and* (6.5), *belongs to* $\mathbf{V}^{1,1/2}(Q)\times(L^2(0,\infty;\mathbf{V}^5(\Omega))\cap H^{3/2}(0,\infty;\mathbf{V}^2(\Omega)))$ *and we have*:

$$\begin{aligned}&\|\hat{\mathbf{y}}\|_{C_b([0,\infty);\mathbf{V}^0_n(\Omega))}+\|\hat{\mathbf{y}}\|_{\mathbf{V}^{1,1/2}(Q_\infty)}+\|\hat{\Phi}\|_{L^2(0,\infty;\mathbf{V}^5(\Omega))}+\|\hat{\Phi}\|_{H^{3/2}(0,\infty;\mathbf{V}^2(\Omega))}\\&\quad\leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}+\|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}^0_n(\Omega))}+\|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}(\Omega))')}\big).\end{aligned}$$

If moreover $\mathbf{y}_0\in\mathbf{V}^{1/2+\varepsilon}_0(\Omega)$, *then* $(\hat{\mathbf{y}},\hat{\Phi})$ *belongs to* $\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q)\times(L^2(0,\infty;\mathbf{V}^{11/2+\varepsilon}(\Omega))\cap H^{7/4+\varepsilon}(0,\infty;\mathbf{V}^2(\Omega)))$, *we have*:

$$\begin{aligned}&\|\hat{\mathbf{y}}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q)}+\|\hat{\Phi}\|_{L^2(0,\infty;\mathbf{V}^{11/2+\varepsilon}(\Omega))}+\|\hat{\Phi}\|_{H^{7/4+\varepsilon}(0,\infty;\mathbf{V}^2(\Omega))}\\&\quad\leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}^{1/2+\varepsilon}_0(\Omega)}+\|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}^0_n(\Omega))}+\|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}(\Omega))')}\big),\end{aligned}$$

and

$$\|B^*\hat{\Phi}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)}\leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}^{1/2+\varepsilon}_0(\Omega)}+\|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}^0_n(\Omega))}+\|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}(\Omega))')}\big).$$

Proof. Assume that $\mathbf{y}_0\in\mathbf{V}^0_n(\Omega)$. Applying Lemmas A.1–A.3, we first obtain that $\hat{\mathbf{y}}$ belongs to $\mathbf{V}^{1/2-\varepsilon',1/4-\varepsilon'/2}(Q_\infty)$ for all $\varepsilon'>0$. From Lemma A.7, we deduce that $B^*\hat{\Phi}\in\mathbf{V}^{1/2-\varepsilon',1/4-\varepsilon'/2}(\Sigma_\infty)$ for all $\varepsilon'>0$. We can use a bootstrap argument to show that $\hat{\mathbf{y}}\in\mathbf{V}^{1,1/2}(Q_\infty)\cap C_b([0,\infty);\mathbf{V}^0_n(\Omega))$ and $\hat{\Phi}\in(L^2(0,\infty;\mathbf{V}^5(\Omega))\cap H^{3/2}(0,\infty;\mathbf{V}^2(\Omega)))$, with the corresponding estimates for $\hat{\mathbf{y}}$ and $\hat{\Phi}$.

Now assume that $\mathbf{y}_0\in\mathbf{V}^{1/2+\varepsilon}_0(\Omega)$. From the estimate proved in the case when $\mathbf{y}_0\in\mathbf{V}^0_n(\Omega)$, we deduce that $\theta^2MB^*\hat{\Phi}$ belongs to $\mathbf{V}^{3,3/2}(\Sigma_\infty)$. Moreover $\theta^2MB^*\hat{\Phi}|_{t=0}=0$. Thus the estimate for $\hat{\mathbf{y}}$ follows from Lemmas A.1–A.3. The estimate for $\hat{\Phi}$ follows from Lemma A.5. □

Lemma 6.3. *For all* $0<\varepsilon\leqslant 1/2$, *all* $\mathbf{y}_0\in\mathbf{V}^{1/2+\varepsilon}_0(\Omega)$, *and all* $\mathbf{f}\in L^1(0,\infty;\mathbf{V}^0_n(\Omega))\cap L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}_n(\Omega))')$, *the solution to the equation*

$$\mathbf{y}'=A_\Pi(t)\mathbf{y}+\mathbf{f},\quad \mathbf{y}(0)=\mathbf{y}_0,\tag{6.11}$$

obeys

$$\|\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)}\leqslant C_1\big(|\mathbf{y}_0|_{\mathbf{V}^{1/2+\varepsilon}_0(\Omega)}+\|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}^0_n(\Omega))}+\|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}_n(\Omega))')}\big).$$

(*The constant* C_1 *depends on* $0<\varepsilon\leqslant 1/2$.)

Proof. *Step 1.* Since the evolution operator $(G(t,s))_{0\leqslant s<t<\infty}$, generated by $A_\Pi(t)$, is exponentially stable on $\mathbf{V}^0_n(\Omega)$, the solution $\mathbf{y}$ to Eq. (6.11) obeys

$$\mathbf{y}(t)=G(t,0)\mathbf{y}_0+\int_0^t G(t,\tau)\mathbf{f}(\tau)\,\mathrm{d}\tau,$$

and

$$\big|\mathbf{y}(t)\big|_{\mathbf{V}^0_n(\Omega)} \leqslant M_1 \mathrm{e}^{-\omega t}|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)} + \int_0^t M_1 \mathrm{e}^{-\omega(t-\tau)}\big|\mathbf{f}(\tau)\big|_{\mathbf{V}^0_n(\Omega)}\,\mathrm{d}\tau.$$

Therefore $\mathbf{y}$ belongs to $C_b([0,\infty);\mathbf{V}^0_n(\Omega)) \cap L^2(0,\infty;\mathbf{V}^0_n(\Omega))$ and

$$\|\mathbf{y}\|_{L^\infty(0,\infty;\mathbf{V}^0_n(\Omega))} + \|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))} \leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}^0_n(\Omega))}\big).$$

We denote by $(\hat{\mathbf{y}}, \hat{\Phi})$ the pair obeying systems (6.4) and (6.5). We set

$$\mathbf{r}(t) = \hat{\Phi}(t) - \Pi(t)\mathbf{y}(t).$$

We denote by $\tilde{\mathbf{y}}$ the solution to the equation

$$\tilde{\mathbf{y}}' = A\tilde{\mathbf{y}} + \theta^2 BM^2B^*\mathbf{r}, \quad \tilde{\mathbf{y}}(0) = 0.$$

We can easily verify that

$$\mathbf{y} = \hat{\mathbf{y}} + \tilde{\mathbf{y}}.$$

Due to Theorem 6.2, we notice that

$$\begin{aligned}
&\big\|\hat{\mathbf{y}}\big\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}^{1/2+\varepsilon}_0(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}^0_n(\Omega))} + \|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}_n(\Omega))')}\big),\\
&\big\|B^*\hat{\Phi}\big\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(\Sigma_\infty)} \leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}^{1/2+\varepsilon}_0(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}^0_n(\Omega))\cap L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}_n(\Omega))')}\big).
\end{aligned} \tag{6.12}$$

Moreover $\mathbf{y}$ belongs to $L^2(0,\infty;\mathbf{V}^0_n(\Omega)) \cap C_b([0,\infty);\mathbf{V}^0_n(\Omega))$ and

$$\big\|B^*\Pi(t)\big\|_{\mathcal{L}(\mathbf{V}^0_n(\Omega),\mathbf{V}^{1/2}(\Gamma))} \leqslant C \quad \text{for all } t \geqslant 0.$$

Thus $B^*\mathbf{r} = B^*\hat{\Phi} - B^*\Pi(\cdot)\mathbf{y}$ belongs to $\mathbf{V}^{0,0}(\Sigma_\infty)$ and

$$\big\|B^*\mathbf{r}\big\|_{\mathbf{V}^{0,0}(\Sigma_\infty)} \leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}^{1/2+\varepsilon}_0(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}^0_n(\Omega))} + \|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}_n(\Omega))')}\big).$$

From Lemma A.3, it follows that

$$\big\|\tilde{\mathbf{y}}\big\|_{\mathbf{V}^{1/2-\varepsilon',1/4-\varepsilon'/2}(Q_\infty)} \leqslant C\big(\big\|B^*\mathbf{r}\big\|_{\mathbf{V}^{0,0}(\Sigma)} + \|\tilde{\mathbf{y}}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))}\big) \quad \text{for all } \varepsilon' > 0.$$

With (6.12), we obtain

$$\|\mathbf{y}\|_{\mathbf{V}^{1/2-\varepsilon',1/4-\varepsilon'/2}(Q_\infty)} \leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}^{1/2+\varepsilon}_0(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}^0_n(\Omega))} + \|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}_n(\Omega))')}\big)$$

for all $\varepsilon' > 0$.

Step 2. Let us prove that $B^*\mathbf{r} \in L^2(0,\infty;\mathbf{V}^{3/2+\varepsilon}(\Gamma))$. With Corollary 5.2, we can write

$$\big|B^*\Pi(t)\mathbf{y}(t)\big|_{\mathbf{V}^{3-\varepsilon'}(\Gamma)} \leqslant C(\varepsilon')\big|\mathbf{y}(t)\big|_{\mathbf{V}^{1/2-\varepsilon'}_n(\Omega)} \quad \text{for all } t > 0 \text{ and all } \varepsilon' > 0.$$

Thus

$$\big\|B^*\Pi\mathbf{y}\big\|_{L^2(0,\infty;\mathbf{V}^{3-\varepsilon'}(\Gamma))} \leqslant C(\varepsilon')\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}^{1/2-\varepsilon'}_n(\Omega))}.$$

Since $B^*\mathbf{r} = B^*\hat{\Phi} - B^*\Pi\mathbf{y}$, with (6.12), we have

$$\big\|B^*\mathbf{r}\big\|_{L^2(0,\infty;\mathbf{V}^{3/2+\varepsilon}(\Gamma))} \leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}^{1/2+\varepsilon}_n(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}^0_n(\Omega))} + \|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}_n(\Omega))')}\big).$$

Step 3. Let us prove that $B^*\Pi(\cdot)\mathbf{y}(\cdot)$ belongs to $H^{3/4+\varepsilon/2}(0,\infty;\mathbf{V}^0(\Gamma))$ if $0 < \varepsilon < 1/2$. From Lemma 6.4, it follows that

$$\begin{aligned}
\big\|B^*\Pi\mathbf{y}\big\|_{H^{1/4-\varepsilon'/2}(0,\infty;\mathbf{V}^0(\Gamma))} &\leqslant C\big(\big\|B^*\Pi\mathbf{y}\big\|_{L^2(0,\infty;\mathbf{V}^0(\Gamma))} + \|\mathbf{y}\|_{H^{1/4-\varepsilon'/2}(0,\infty;\mathbf{V}^0_n(\Omega))}\big)\\
&\leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}^{1/2+\varepsilon}_n(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}^0_n(\Omega))} + \|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}_n(\Omega))')}\big)
\end{aligned}$$

for all $\varepsilon' > 0$. With the equality $B^*\mathbf{r} = B^*\hat{\Phi} - B^*\Pi\mathbf{y}$, with (6.12), and with step 2, we have

$$\|B^*\mathbf{r}\|_{\mathbf{V}^{1/2-\varepsilon',1/4-\varepsilon'/2}(\Sigma_\infty)} \leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}_0^{1/2+\varepsilon}(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))} + \|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')}\big),$$

for all $\varepsilon' > 0$. With Lemma A.3, we obtain

$$\|\tilde{\mathbf{y}}\|_{\mathbf{V}^{1-\varepsilon'',1/2-\varepsilon''/2}(Q_\infty)} \leqslant C\big(\|B^*\mathbf{r}\|_{\mathbf{V}^{1/2-\varepsilon',1/4-\varepsilon'/2}(\Sigma)} + \|\tilde{\mathbf{y}}\|_{L^2(0,\infty;\mathbf{V}_n^0(\Omega))}\big),$$

for all $\varepsilon'' > \varepsilon' > 0$. With (6.12), we can claim that $\mathbf{y}$ obeys the estimate

$$\|\mathbf{y}\|_{\mathbf{V}^{1-\varepsilon',1-\varepsilon'/2}(Q_\infty)} \leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}_0^{1/2+\varepsilon}(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))} + \|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')}\big),$$

for all $\varepsilon' > 0$. Reiterating this process, we can show that

$$\|\mathbf{y}\|_{\mathbf{V}^{3/2-\varepsilon',3/4-\varepsilon'/2}(Q_\infty)} \leqslant C\big(|\mathbf{y}_0|_{\mathbf{V}_0^{1/2+\varepsilon}(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))} + \|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')}\big),$$

for all $\varepsilon' > 0$. Another iteration gives

$$\|\mathbf{y}\|_{\mathbf{V}^{2-\varepsilon',1-\varepsilon'/2}(Q_\infty)} \leqslant C\big(\|\mathbf{y}_0\|_{\mathbf{V}_0^{1/2+\varepsilon}(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))} + \|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')}\big),$$

for all $\varepsilon' > 0$.

Applying Lemma 6.4, we can show that $B^*\Pi(\cdot)\mathbf{y}(\cdot)$ belongs to $H^{3/4+\varepsilon/2}(0,\infty;\mathbf{V}^0(\Gamma))$ if $0 < \varepsilon < 1/2$. Still with Lemma A.3, we have

$$\|\tilde{\mathbf{y}}\|_{\mathbf{V}^{2,1}(Q_\infty)} \leqslant C\big(\|B^*\mathbf{r}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(\Sigma)} + \|\tilde{\mathbf{y}}\|_{L^2(0,\infty;\mathbf{V}_n^0(\Omega))}\big),$$

if $0 < \varepsilon < 1/2$. Thus, with (6.12), we finally obtain

$$\|\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \leqslant C\big(\|\mathbf{y}_0\|_{\mathbf{V}_0^{1/2+\varepsilon}(\Omega)} + \|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))} + \|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')}\big),$$

for all $0 < \varepsilon \leqslant 1/2$. The proof is complete. □

Lemma 6.4. *Assume that $\mathbf{y} \in H^\sigma(0,\infty;\mathbf{V}_n^0(\Omega))$ for some $0 < \sigma < 1$, and that $B^*\Pi\mathbf{y} \in L^2(0,\infty;\mathbf{V}^0(\Gamma))$, then $B^*\Pi\mathbf{y} \in H^\sigma(0,\infty;\mathbf{V}^0(\Gamma))$, and*

$$\|B^*\Pi\mathbf{y}\|_{H^\sigma(0,\infty;\mathbf{V}^0(\Gamma))} \leqslant C(\sigma)\big(\|B^*\Pi\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}^0(\Gamma))} + \|\mathbf{y}\|_{H^\sigma(0,\infty;\mathbf{V}_n^0(\Omega))}\big).$$

Proof. To prove Lemma 6.4, we have only to estimate the integral

$$\int_0^\infty\int_0^\infty \frac{|B^*\Pi(t)\mathbf{y}(t) - B^*\Pi(\tau)\mathbf{y}(\tau)|^2_{\mathbf{V}^0(\Gamma)}}{|t-\tau|^{1+2\sigma}}\,\mathrm{d}t\,\mathrm{d}\tau.$$

We have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{|B^*\Pi(t)\mathbf{y}(t) - B^*\Pi(\tau)\mathbf{y}(\tau)|^2_{\mathbf{V}^0(\Gamma)}}{|t-\tau|^{1+2\sigma}}\,\mathrm{d}t\,\mathrm{d}\tau \\
&\quad= \int_0^{2t_0}\int_0^{2t_0} \frac{|B^*\Pi(t)\mathbf{y}(t) - B^*\Pi(\tau)\mathbf{y}(\tau)|^2_{\mathbf{V}^0(\Gamma)}}{|t-\tau|^{1+2\sigma}}\,\mathrm{d}t\,\mathrm{d}\tau + \int_{2t_0}^\infty\int_0^{2t_0} \frac{|B^*\Pi(t)\mathbf{y}(t) - B^*\widehat{\Pi}\mathbf{y}(\tau)|^2_{\mathbf{V}^0(\Gamma)}}{|t-\tau|^{1+2\sigma}}\,\mathrm{d}t\,\mathrm{d}\tau \\
&\qquad+ \int_0^{2t_0}\int_{2t_0}^\infty \frac{|B^*\widehat{\Pi}\mathbf{y}(t) - B^*\Pi(\tau)\mathbf{y}(\tau)|^2_{\mathbf{V}^0(\Gamma)}}{|t-\tau|^{1+2\sigma}}\,\mathrm{d}t\,\mathrm{d}\tau + \int_{2t_0}^\infty\int_{2t_0}^\infty \frac{|B^*\widehat{\Pi}(\mathbf{y}(t) - \mathbf{y}(\tau))|^2_{\mathbf{V}^0(\Gamma)}}{|t-\tau|^{1+2\sigma}}\,\mathrm{d}t\,\mathrm{d}\tau \\
&\quad\leqslant \int_0^{2t_0}\int_0^{2t_0} \frac{|B^*\Pi(t)(\mathbf{y}(t) - \mathbf{y}(\tau))|^2_{\mathbf{V}^0(\Gamma)}}{|t-\tau|^{1+2\sigma}}\,\mathrm{d}t\,\mathrm{d}\tau + \int_0^{2t_0}\int_0^{2t_0} \frac{|(B^*\Pi(t) - B^*\Pi(\tau))\mathbf{y}(\tau)|^2_{\mathbf{V}^0(\Gamma)}}{|t-\tau|^{1+2\sigma}}\,\mathrm{d}t\,\mathrm{d}\tau
\end{aligned}$$

$$+2\int_{2t_0}^{\infty}\int_0^{2t_0}\frac{|B^*\Pi(t)\mathbf{y}(t)-B^*\widehat{\Pi}\mathbf{y}(\tau)|^2_{\mathbf{V}^0(\Gamma)}}{|t-\tau|^{1+2\sigma}}\,\mathrm{d}t\,\mathrm{d}\tau+\|B^*\widehat{\Pi}\|^2_{\mathcal{L}(\mathbf{V}^0_n(\Omega),\mathbf{V}^0(\Gamma))}\|\mathbf{y}\|^2_{H^\sigma(0,\infty;\mathbf{V}^0_n(\Omega))}.$$

Let us examine the different terms. We have

$$\int_0^{2t_0}\int_0^{2t_0}\frac{|B^*\Pi(t)(\mathbf{y}(t)-\mathbf{y}(\tau))|^2_{\mathbf{V}^0(\Gamma)}}{|t-\tau|^{1+2\sigma}}\,\mathrm{d}t\,\mathrm{d}\tau\leqslant\sup_{s\geqslant 0}\big\|B^*\Pi(s)\big\|^2_{\mathcal{L}(\mathbf{V}^0_n(\Omega),\mathbf{V}^0(\Gamma))}\|\mathbf{y}\|^2_{H^\sigma(0,\infty;\mathbf{V}^0_n(\Omega))}.$$

With Corollary 5.4 , we can write

$$\begin{aligned}\int_0^{2t_0}\int_0^{2t_0}\frac{|(B^*\Pi(t)-B^*\Pi(\tau))\mathbf{y}(\tau)|^2_{\mathbf{V}^0(\Gamma)}}{|t-\tau|^{1+2\sigma}}\,\mathrm{d}t\,\mathrm{d}\tau&\leqslant C\int_0^{2t_0}\left(\big|\mathbf{y}(\tau)\big|^2_{\mathbf{V}^0_n(\Omega)}\int_0^{2t_0}\frac{1}{|t-\tau|^{2\sigma-1}}\,\mathrm{d}t\right)\mathrm{d}\tau\\&\leqslant C\|\mathbf{y}\|^2_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))}.\end{aligned}$$

Moreover

$$\begin{aligned}&\int_{2t_0}^{\infty}\int_0^{2t_0}\frac{|B^*\Pi(t)\mathbf{y}(t)-B^*\widehat{\Pi}\mathbf{y}(\tau)|^2_{\mathbf{V}^0(\Gamma)}}{|t-\tau|^{1+2\sigma}}\,\mathrm{d}t\,\mathrm{d}\tau\\&\quad=\int_0^{t_0}\int_{2t_0}^{\infty}\frac{|B^*\Pi(t)\mathbf{y}(t)-B^*\widehat{\Pi}\mathbf{y}(\tau)|^2_{\mathbf{V}^0(\Gamma)}}{|t-\tau|^{1+2\sigma}}\,\mathrm{d}\tau\,\mathrm{d}t+\int_{t_0}^{2t_0}\int_{2t_0}^{\infty}\frac{|B^*\widehat{\Pi}(\mathbf{y}(t)-\mathbf{y}(\tau))|^2_{\mathbf{V}^0(\Gamma)}}{|t-\tau|^{1+2\sigma}}\,\mathrm{d}\tau\,\mathrm{d}t\\&\quad\leqslant\frac{4}{|t_0|^{2\sigma}}\sup_{s\geqslant 0}\big\|B^*\Pi(s)\big\|^2_{\mathcal{L}(\mathbf{V}^0_n(\Omega),\mathbf{V}^0(\Gamma))}\|\mathbf{y}\|^2_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))}\\&\qquad+\big\|B^*\widehat{\Pi}\big\|^2_{\mathcal{L}(\mathbf{V}^0_n(\Omega),\mathbf{V}^0(\Gamma))}\|\mathbf{y}\|^2_{H^\sigma(0,\infty;\mathbf{V}^0_n(\Omega))}.\end{aligned}$$

We complete the proof by combining the previous estimates. □

7. Stabilization of the three-dimensional Navier–Stokes equations

7.1. First stabilization result

Consider the Navier–Stokes equations with the linear feedback law associated with $A_\Pi(\cdot)$:

$$\begin{aligned}&P\mathbf{y}'=A_\Pi(t)P\mathbf{y}+F(\mathbf{y}),\quad P\mathbf{y}(0)=P\mathbf{y}_0,\\&(I-P)\mathbf{y}=-(I-P)D_A\theta^2(t)M^2B_n^*\Pi(t)P\mathbf{y},\end{aligned}\tag{7.1}$$

where $F(\mathbf{y})=-P[(\mathbf{y}\cdot\nabla)\mathbf{y}]=-P[\operatorname{div}(\mathbf{y}\otimes\mathbf{y})]$. System (7.1) is clearly a closed-loop system with a feedback control pointwise in time. We can give different equivalent formulations of system (7.1). For that, let us introduce the pressure $\psi(t)$ associated with $\Pi(t)P\mathbf{y}(t)$, that is the function $\psi(t)$ solution to the elliptic equation

$$\begin{aligned}&\Delta\psi(t)=\operatorname{div}\big(-(\nabla\mathbf{w})^T\big(\Pi(t)P\mathbf{y}\big)+(\mathbf{w}\cdot\nabla)\big(\Pi(t)P\mathbf{y}\big)\big)\quad\text{in }\Omega,\\&\frac{\partial\psi(t)}{\partial\mathbf{n}}=\big(\nu\Delta\big(\Pi(t)P\mathbf{y}\big)-(\nabla\mathbf{w})^T\big(\Pi(t)P\mathbf{y}\big)+(\mathbf{w}\cdot\nabla)\big(\Pi(t)P\mathbf{y}\big)\big)\cdot\mathbf{n}\quad\text{on }\Gamma.\end{aligned}\tag{7.2}$$

We also introduce an auxiliary function $q(t)$ solution to the elliptic equation

$$\Delta q(t)=0\text{ in }\Omega,\quad\frac{\partial q(t)}{\partial\mathbf{n}}=-\theta^2(t)\big(M^2\big(\psi(t)\mathbf{n}-c\big(\psi(t)\big)\mathbf{n}\big)\big)\cdot\mathbf{n}\text{ on }\Gamma.\tag{7.3}$$

System (7.1) can be rewritten in the following form:

$$\begin{aligned}
&\frac{\partial \mathbf{y}}{\partial t}-\nu\Delta\mathbf{y}+(\mathbf{y}\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{y}+(\mathbf{y}\cdot\nabla)\mathbf{y}+\nabla p=0,\quad \operatorname{div}\mathbf{y}=0\quad\text{in } Q_\infty,\\
&\mathbf{y}=\theta^2(t)M^2\left(\nu\frac{\partial(\Pi(t)P\mathbf{y})}{\partial\mathbf{n}}-\psi\mathbf{n}+c(\psi)\mathbf{n}\right)\text{ on }\Sigma_\infty,\quad \mathbf{y}(0)=\mathbf{y}_0\text{ in }\Omega,
\end{aligned}\tag{7.4}$$

where $\psi(t)$ is the solution of (7.2). Another equivalent formulation is the following one:

$$\begin{aligned}
&\frac{\partial P\mathbf{y}}{\partial t}-\nu\Delta P\mathbf{y}+(P\mathbf{y}\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)P\mathbf{y}+(\mathbf{y}\cdot\nabla)\mathbf{y}+\nabla r=0\quad\text{in } Q_\infty,\\
&P\mathbf{y}=\theta^2(t)m^2\nu\frac{\partial(\Pi(t)P\mathbf{y})}{\partial\mathbf{n}}-\nabla_\tau q(t),\text{ on }\Sigma_\infty,\quad P\mathbf{y}(0)=P\mathbf{y}_0\text{ in }\Omega,\quad\text{and}\\
&(I-P)\mathbf{y}=-(I-P)D_A\theta^2(t)M^2\big(\psi(t)\mathbf{n}-c\big(\psi(t)\big)\mathbf{n}\big),
\end{aligned}\tag{7.5}$$

where $\psi(t)$ is the solution of (7.2), $q(t)$ is the solution of (7.3), and ∇_τ denotes the tangential gradient operator. The equivalence between the systems (7.1), (7.4), and (7.5) follows from calculations in [18]. Observe that in system (7.5) the condition $\operatorname{div}\mathbf{y}=0$ is satisfied because $\operatorname{div}(P\mathbf{y})=0$ and $\operatorname{div}((I-P)\mathbf{y})=0$. This last identity follows from the equality $\operatorname{div}(D_A\theta^2(t)M^2(\psi(t)\mathbf{n}-c(\psi(t))\mathbf{n}))=0$, from the definition of $(I-P)\mathbf{y}$ and the one of the operator $(I-P)$.

Theorem 7.1. *For all $0<\varepsilon\leqslant 1/2$, there exist $\mu_0=\mu_0(\mathbf{w},\varepsilon)>0$ and $\bar C_0(\mathbf{w},\varepsilon)>0$, such that if $\mu\in[0,\mu_0]$ and $|\mathbf{y}_0|_{\mathbf{V}_0^{1/2+\varepsilon}(\Omega)}\leqslant \bar C_0(\mathbf{w},\varepsilon)\mu$, then Eq. (7.1) admits a unique solution $\mathbf{y}$ such that $P\mathbf{y}\in D_\mu$, where*

$$D_\mu=\big\{P\mathbf{y}\in\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)\ \big|\ \|P\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)}\leqslant\mu\big\}.$$

Moreover $\mathbf{y}$ belongs to $C_b([0,\infty);\mathbf{V}^{1/2+\varepsilon}(\Omega))$ and it satisfies

$$\big|\mathbf{y}(t)\big|_{\mathbf{V}^{1/2+\varepsilon}(\Omega)}\leqslant\bar C_1(\mathbf{w},\varepsilon)\mu\quad\textit{for all } t\geqslant 0.$$

Lemma 7.1. *If $\mathbf{z}$ and $\mathbf{y}$ belong to $\mathbf{V}^{3/2+\varepsilon/2,3/4+\varepsilon/4}(Q_\infty)$, with $0<\varepsilon\leqslant 1/2$, then*

$$\begin{aligned}
&\big\|P\operatorname{div}(\mathbf{z}\otimes\mathbf{y})\big\|_{L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')}+\big\|P\operatorname{div}(\mathbf{z}\otimes\mathbf{y})\big\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))}\\
&\quad\leqslant C_2\|\mathbf{z}\|_{\mathbf{V}^{3/2+\varepsilon/2,3/4+\varepsilon/4}(Q_\infty)}\|\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon/2,3/4+\varepsilon/4}(Q_\infty)}.
\end{aligned}$$

Proof. If $\mathbf{z}$ and $\mathbf{y}$ belong to $\mathbf{V}^{3/2+\varepsilon/2}(\Omega)$, then

$$|\mathbf{z}\otimes\mathbf{y}|_{(\mathbf{H}^1(\Omega))^3}\leqslant|\mathbf{z}\otimes\mathbf{y}|_{(\mathbf{H}^{3/2}(\Omega))^3}\leqslant C|\mathbf{z}|_{\mathbf{V}^{3/2+\varepsilon/2}(\Omega)}|\mathbf{y}|_{\mathbf{V}^{3/2+\varepsilon/2}(\Omega)}.$$

(See e.g. [15, Proposition B.1].) Thus we have

$$\big\|P\operatorname{div}(\mathbf{z}\otimes\mathbf{y})\big\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))}\leqslant C\|\mathbf{z}\|_{L^2(0,\infty;\mathbf{V}^{3/2+\varepsilon}(\Omega))}\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}^{3/2+\varepsilon}(\Omega))}.$$

We also have

$$\|\mathbf{z}\|_{L^4(0,\infty;\mathbf{V}^{1+\varepsilon/2}(\Omega))}\leqslant C\|\mathbf{z}\|_{H^{1/4}(0,\infty;\mathbf{V}^{1+\varepsilon/2}(\Omega))}\leqslant C\|\mathbf{z}\|^{(2+\varepsilon)/(3+\varepsilon)}_{L^2(0,\infty;\mathbf{V}^{3/2+\varepsilon/2}(\Omega))}\|\mathbf{z}\|^{1/(3+\varepsilon)}_{H^{3/4+\varepsilon/4}(0,\infty;\mathbf{V}^0(\Omega))}.$$

Still with [15, Proposition B.1], we can write

$$\begin{aligned}
\|\mathbf{z}\otimes\mathbf{y}\|_{(L^2(0,\infty;\mathbf{H}^{1/2+\varepsilon}(\Omega)))^3}&\leqslant C\|\mathbf{z}\|_{L^4(0,\infty;\mathbf{V}^{1+\varepsilon/2}(\Omega))}\|\mathbf{y}\|_{L^4(0,\infty;\mathbf{V}^{1+\varepsilon/2}(\Omega))}\\
&\leqslant C\|\mathbf{z}\|^{(2+\varepsilon)/(3+\varepsilon)}_{L^2(0,\infty;\mathbf{V}^{3/2+\varepsilon/2}(\Omega))}\|\mathbf{y}\|^{(2+\varepsilon)/(3+\varepsilon)}_{L^2(0,\infty;\mathbf{V}^{3/2+\varepsilon/2}(\Omega))}\\
&\quad\times\|\mathbf{z}\|^{1/(3+\varepsilon)}_{H^{3/4+\varepsilon/4}(0,\infty;\mathbf{V}^0(\Omega))}\|\mathbf{y}\|^{1/(3+\varepsilon)}_{H^{3/4+\varepsilon/4}(0,\infty;\mathbf{V}^0(\Omega))},
\end{aligned}$$

for all $0<\varepsilon\leqslant 1/2$. The mapping $\zeta\mapsto P\operatorname{div}\zeta$ is continuous from $(\mathbf{H}^1(\Omega))^3$ into $\mathbf{V}_n^0(\Omega)$. Indeed $\operatorname{div}\zeta$ can be identified with the mapping

$$\Phi\mapsto\int_\Omega\operatorname{div}\zeta\cdot\Phi.$$

The mapping $\zeta \mapsto P \operatorname{div} \zeta$ can be extended to a continuous operator from $(\mathbf{L}^2(\Omega))^3$ into $\mathcal{V}^{-1}(\Omega)$ by the formula

$$P \operatorname{div} \zeta : \Phi \mapsto \int_\Omega \zeta \cdot \nabla \Phi \quad \text{for all } \Phi \in \mathbf{V}_0^1(\Omega).$$

Thus, by interpolation, if $\mathbf{z} \otimes \mathbf{y}$ belongs to $(\mathbf{H}^{1/2+\varepsilon}(\Omega))^3$, then $P \operatorname{div}(\mathbf{z} \otimes \mathbf{y})$ can be identified with an element in $(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))'$, and we have:

$$\begin{aligned}
&\left\| P \operatorname{div}(\mathbf{z} \otimes \mathbf{y}) \right\|_{L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')} \\
&\quad \leqslant C \|\mathbf{z}\|_{L^2(0,\infty;\mathbf{V}^{3/2+\varepsilon/2}(\Omega))}^{(2+\varepsilon)/(3+\varepsilon)} \|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}^{3/2+\varepsilon/2}(\Omega))}^{(2+\varepsilon)/(3+\varepsilon)} \|\mathbf{z}\|_{H^{3/4+\varepsilon/4}(0,\infty;\mathbf{V}^0(\Omega))}^{1/(3+\varepsilon)} \|\mathbf{y}\|_{H^{3/4+\varepsilon/4}(0,\infty;\mathbf{V}^0(\Omega))}^{1/(3+\varepsilon)} \\
&\quad \leqslant C \|\mathbf{z}\|_{\mathbf{V}^{3/2+\varepsilon/2,3/4+\varepsilon/4}(Q_\infty)} \|\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon/2,3/4+\varepsilon/4}(Q_\infty)}. \qquad \square
\end{aligned}$$

Lemma 7.2. *If* $\mathbf{y}$ *belongs to* $\mathbf{V}^{3/2+\varepsilon/2,3/4+\varepsilon/4}(Q_\infty)$ *for some* $0 < \varepsilon \leqslant 1/2$*, then*

$$\left\| (I-P)\theta^2(\cdot) D_A M^2 B^* \Pi(\cdot) P\mathbf{y} \right\|_{\mathbf{V}^{3/2+\varepsilon/2,3/4+\varepsilon/4}(Q_\infty)} \leqslant C_3 \|\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon/2,3/4+\varepsilon/4}(Q_\infty)},$$

and

$$\left\| (I-P)\theta^2(\cdot) D_A M^2 B^* \Pi(\cdot) P\mathbf{y} \right\|_{C_b([0,\infty);\mathbf{V}^{1/2+\varepsilon}(\Omega))} \leqslant C_4 \|P\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon/2,3/4+\varepsilon/4}(Q_\infty)}.$$

Proof. *Step 1.* For all $t \geqslant 0$, we have the uniform estimate

$$\left\| B^* \Pi(t) \right\|_{\mathcal{L}(\mathbf{V}_n^{1/2-\varepsilon}(\Omega), \mathbf{V}^{3-\varepsilon}(\Gamma))} \leqslant C \quad \text{for all } t \geqslant 0$$

(see Corollary 5.2). Thus

$$\left\| (I-P)\theta^2(t) D_A M^2 B^* \Pi(t) \right\|_{\mathcal{L}(\mathbf{V}_n^{1/2-\varepsilon}(\Omega), \mathbf{V}^{7/2-\varepsilon}(\Omega))} \leqslant C \quad \text{for all } t > 0.$$

Therefore we have

$$\left\| (I-P)\theta^2(\cdot) D_A M^2 B^* \Pi(\cdot) P\mathbf{y} \right\|_{L^2(0,\infty;\mathbf{V}^{7/2-\varepsilon}(\Omega))} \leqslant C \|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}_n^{1/2-\varepsilon}(\Omega))}.$$

Step 2. We are going to show that

$$\left\| (I-P)\theta^2(\cdot) D_A M^2 B^* \Pi(\cdot) P\mathbf{y} \right\|_{H^{3/4+\varepsilon/4}(0,\infty;\mathbf{V}^0(\Omega))} \leqslant C \|\mathbf{y}\|_{H^{3/4+\varepsilon/4}(0,\infty;\mathbf{V}_n^0(\Omega))}.$$

We have

$$\begin{aligned}
&\left\| (I-P)\theta^2(\cdot) D_A M^2 B^* \Pi(\cdot) P\mathbf{y} \right\|_{H^{3/4+\varepsilon/4}(0,\infty;\mathbf{V}^0(\Omega))}^2 \\
&\quad \leqslant C \left\| (I-P)\theta^2(\cdot) D_A M^2 B^* \Pi(\cdot) P\mathbf{y} \right\|_{L^2(0,\infty;\mathbf{V}^0(\Omega))}^2 \\
&\qquad + C \int_0^\infty \int_0^\infty \frac{|B^* \Pi(t) P \theta^2(t) \mathbf{y}(t) - B^* \Pi(\tau) P \theta^2(\tau) \mathbf{y}(\tau))|_{\mathbf{V}^0(\Gamma)}^2}{|t-\tau|^{5/2+\varepsilon/2}} \, \mathrm{d}t \, \mathrm{d}\tau.
\end{aligned}$$

As in Lemma 6.4, we can show that

$$\int_0^\infty \int_0^\infty \frac{|B^* \Pi(t) P \theta^2(t) \mathbf{y}(t) - B^* \Pi(\tau) P \theta^2(\tau) \mathbf{y}(\tau))|_{\mathbf{V}^0(\Gamma)}^2}{|t-\tau|^{5/2+\varepsilon/2}} \, \mathrm{d}t \, \mathrm{d}\tau \leqslant C \|\mathbf{y}\|_{H^{3/4+\varepsilon/4}(0,\infty;\mathbf{V}^0(\Omega))}.$$

Step 3. From steps 1 and 2, we deduce

$$\left\| (I-P)\theta^2(\cdot) D_A M^2 B^* \Pi(\cdot) P\mathbf{y} \right\|_{\mathbf{V}^{7/2-\varepsilon,3/4+\varepsilon/4}(Q_\infty)} \leqslant C \|\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon/2,3/4+\varepsilon/4}(Q_\infty)}.$$

Since $\mathbf{V}^{7/2-\varepsilon,3/4+\varepsilon/4}(Q_\infty) \hookrightarrow \mathbf{V}^{3/2+\varepsilon/2,3/4+\varepsilon/4}(Q_\infty)$ and $\mathbf{V}^{7/2-\varepsilon,3/4+\varepsilon/4}(Q_\infty) \hookrightarrow C_b([0,\infty);\mathbf{V}^{1/2+\varepsilon}(\Omega))$, the proof is complete. $\square$

We set:

$$\mathcal{F}(\mathbf{z},\mathbf{y}) = \operatorname{div}\big[\big(P\mathbf{z} - (I-P)\theta^2(\cdot)D_A M^2 B_n^* \Pi(\cdot)P\mathbf{z}\big) \otimes \big(P\mathbf{y} - (I-P)\theta^2(\cdot)D_A M^2 B_n^* \Pi(\cdot)P\mathbf{y}\big)\big].$$

We have

$$P\mathcal{F}(\mathbf{z},\mathbf{z}) = F\big(P\mathbf{z} - (I-P)\theta^2(\cdot)D_A M^2 B_n^* \Pi(\cdot)P\mathbf{z}\big).$$

Lemma 7.3. *There exists a constant C_2' such that*

$$\big\|P\mathcal{F}(\mathbf{z},\mathbf{y})\big\|_{L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')\cap L^1(0,\infty;\mathbf{V}_n^0(\Omega))} \leqslant C_2' \|\mathbf{z}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \|\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)},$$

for all $\mathbf{z}$ and $\mathbf{y}$ belonging to $\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)$.

Proof. With Lemmas 7.1 and 7.2, we can write

$$\begin{aligned}
&\big\|P\mathcal{F}(\mathbf{z},\mathbf{y})\big\|_{L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')\cap L^1(0,\infty;\mathbf{V}_n^0(\Omega))} \\
&\quad\leqslant \big\|P\operatorname{div}(P\mathbf{z}\otimes P\mathbf{y})\big\|_{L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')\cap L^1(0,\infty;\mathbf{V}_n^0(\Omega))} \\
&\qquad + \big\|P\operatorname{div}\big(P\mathbf{z}\otimes (I-P)\theta^2(\cdot)D_A M^2 B_n^* \Pi(\cdot)P\mathbf{y}\big)\big\|_{L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')\cap L^1(0,\infty;\mathbf{V}_n^0(\Omega))} \\
&\qquad + \big\|P\operatorname{div}\big((I-P)\theta^2(\cdot)D_A M^2 B_n^* \Pi(\cdot)P\mathbf{z}\otimes P\mathbf{y}\big)\big\|_{L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')\cap L^1(0,\infty;\mathbf{V}_n^0(\Omega))} \\
&\qquad + \big\|P\operatorname{div}\big((I-P)\theta^2 D_A M^2 B_n^* \Pi P\mathbf{z}\otimes (I-P)\theta^2 D_A M^2 B_n^* \Pi P\mathbf{y}\big)\big\|_{L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')\cap L^1(0,\infty;\mathbf{V}_n^0(\Omega))} \\
&\quad\leqslant C_2(1+C_3)^2 \|\mathbf{z}\|_{\mathbf{V}^{3/2+\varepsilon/2,3/4+\varepsilon/4}(Q_\infty)} \|\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon/2,3/4+\varepsilon/4}(Q_\infty)} \\
&\quad\leqslant C_2' \|\mathbf{z}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \|\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)}. \qquad \square
\end{aligned}$$

Lemma 7.4. *For all $0 < \varepsilon \leqslant 1/2$, the mapping $\mathbf{z} \mapsto P\mathcal{F}(\mathbf{z},\mathbf{z})$ is locally Lipschitz continuous from $\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)$ into $L^1(0,\infty;\mathbf{V}_n^0(\Omega)) \cap L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}(\Omega))')$, more precisely we have*

$$\begin{aligned}
&\big\|P\mathcal{F}(\mathbf{z}_1,\mathbf{z}_1) - P\mathcal{F}(\mathbf{z}_2,\mathbf{z}_2)\big\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))} + \big\|P\mathcal{F}(\mathbf{z}_1,\mathbf{z}_1) - P\mathcal{F}(\mathbf{z}_2,\mathbf{z}_2)\big\|_{L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}(\Omega))')} \\
&\quad \leqslant C_2'\big(\|\mathbf{z}_1\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} + \|\mathbf{z}_2\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)}\big) \|\mathbf{z}_1 - \mathbf{z}_2\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)}.
\end{aligned}$$

Proof. By calculations similar to those in Lemma 7.3, we have:

$$\begin{aligned}
&\big\|P\mathcal{F}(\mathbf{z}_1,\mathbf{z}_1) - P\mathcal{F}(\mathbf{z}_2,\mathbf{z}_2)\big\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))\cap L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}(\Omega))')} \\
&\quad \leqslant \big\|P\mathcal{F}(\mathbf{z}_1,\mathbf{z}_1-\mathbf{z}_2)\big\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))\cap L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}(\Omega))')} \\
&\qquad + \big\|P\mathcal{F}(\mathbf{z}_1-\mathbf{z}_2,\mathbf{z}_2)\big\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))\cap L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}(\Omega))')} \\
&\quad \leqslant C_2' \|\mathbf{z}_1\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \|\mathbf{z}_1-\mathbf{z}_2\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \\
&\qquad + C_2' \|\mathbf{z}_1-\mathbf{z}_2\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \|\mathbf{z}_2\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)}.
\end{aligned}$$

The proof is complete. $\square$

Proof of Theorem 7.1. We set $\mu_0 = \frac{1}{4C_1C_2'}$ and $\bar{C}_0(\mathbf{w},\varepsilon) = \frac{3}{4C_1}$. ($C_1$ is the continuity constant appearing in Lemma 6.3, and it clearly depends on $\mathbf{w}$ and on ε. The constant C_2' is the one appearing in Lemma 7.3, and it depends on ε.) For $\mathbf{z} \in \mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)$, we denote by $P\mathbf{y}_\mathbf{z}$ the solution to the equation

$$P\mathbf{y}' = A_\Pi(t)P\mathbf{y} + P\mathcal{F}(\mathbf{z},\mathbf{z}), \quad \mathbf{y}(0) = \mathbf{y}_0. \tag{7.6}$$

We are going to prove that the mapping $\mathbf{M}: \mathbf{z} \mapsto P\mathbf{y}_\mathbf{z}$ is a contraction in D_μ if $|\mathbf{y}_0|_{\mathbf{V}_0^{1/2+\varepsilon}} \leqslant \bar{C}_0(\mathbf{w},\varepsilon)\mu$.

Step 1. From Lemmas 7.3 and 6.3 it follows that

$$\|P\mathbf{y_z}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \leqslant C_1\big(|\mathbf{y}_0|_{\mathbf{V}_0^{1/2+\varepsilon}(\Omega)} + \big\|P\mathcal{F}(\mathbf{z},\mathbf{z})\big\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))\cap L^2(0,\infty;(\mathbf{V}^{1/2-2\varepsilon}(\Omega))')}\big)$$
$$\leqslant C_1\left(\frac{3\mu}{4C_1} + C_2'\|\mathbf{z}\|^2_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)}\right) \leqslant \frac{3\mu}{4} + C_1C_2'\mu^2 \leqslant \mu.$$

Thus $\mathbf{M}$ is a mapping from D_μ into itself.

Step 2. From Lemmas 6.3, 7.4 and 7.2, it follows that

$$\begin{aligned}
&\|P\mathbf{y}_{\mathbf{z}_1} - P\mathbf{y}_{\mathbf{z}_2}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \\
&\quad\leqslant C_1\big\|P\mathcal{F}(\mathbf{z}_1,\mathbf{z}_1) - P\mathcal{F}(\mathbf{z}_2,\mathbf{z}_2)\big\|_{L^1(0,\infty;\mathbf{L}^2(\Omega))\cap L^2(0,\infty;(\mathbf{V}^{1/2-\varepsilon}(\Omega))')} \\
&\quad\leqslant C_1C_2'\big(\|\mathbf{z}_1\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} + \|\mathbf{z}_2\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)}\big)\|\mathbf{z}_1 - \mathbf{z}_2\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \\
&\quad\leqslant 2C_1C_2'\mu\|\mathbf{z}_1 - \mathbf{z}_2\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \leqslant \frac{1}{2}\|\mathbf{z}_1 - \mathbf{z}_2\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)}.
\end{aligned}$$

Thus if $\mu \leqslant \mu_0$, the mapping $\mathbf{M}$ is a contraction in D_μ, and the system (7.1) admits a unique solution $\mathbf{y}$ such that $P\mathbf{y}$ belongs to D_μ.

Step 2. From Lemma 7.2, it follows that

$$\begin{aligned}
\big|\mathbf{y}(t)\big|_{\mathbf{V}^{1/2+\varepsilon}(\Omega)} &\leqslant \big|(I-P)\mathbf{y}(t)\big|_{\mathbf{V}^{1/2+\varepsilon}(\Omega)} + \big|P\mathbf{y}(t)\big|_{\mathbf{V}^{1/2+\varepsilon}(\Omega)} \\
&\leqslant C_4\|P\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon/2,3/4+\varepsilon/4}(Q_\infty)} + C_i\|P\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \\
&\leqslant \bar{C}_1(\mathbf{w},\varepsilon)\mu \quad \text{for all } t \geqslant 0,
\end{aligned}$$

where C_i is the continuity constant of the imbedding from $\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)$ into $L^\infty(0,\infty;\mathbf{V}^{1/2+\varepsilon}(\Omega))$. The proof is complete. □

7.2. *Second stabilization result*

As in [19], we can obtain a local exponential stabilization of the Navier–Stokes, with a prescribed decay rate $-\omega < 0$. For that, we set

$$\hat{\mathbf{y}} = \mathrm{e}^{\omega t}\mathbf{y}, \qquad \hat{\mathbf{u}} = \mathrm{e}^{\omega t}\mathbf{u}.$$

If

$$\begin{aligned}
&P\mathbf{y}' = AP\mathbf{y} + PF(\mathbf{y}) + \theta BM\mathbf{u}, \quad P\mathbf{y}(0) = \mathbf{y}_0, \\
&(I-P)\mathbf{y} = (I-P)D_A\theta\gamma_n M\mathbf{u},
\end{aligned}$$

then $\hat{\mathbf{y}}$ is the solution to the system

$$\begin{aligned}
&P\hat{\mathbf{y}}' = AP\hat{\mathbf{y}} + \omega P\hat{\mathbf{y}} + \mathrm{e}^{-\omega t}PF(\hat{\mathbf{y}}) + \theta BM\hat{\mathbf{u}}, \quad P\hat{\mathbf{y}}(0) = \mathbf{y}_0, \\
&(I-P)\hat{\mathbf{y}} = (I-P)D_A\theta\gamma_n M\hat{\mathbf{u}}.
\end{aligned} \tag{7.7}$$

Set $A_\omega = A + \omega I$, and let $\Pi_\omega \in C_s([0,\infty);\mathcal{L}(\mathbf{V}_n^0(\Omega)))$ be the solution to system

$$\begin{aligned}
&\Pi_\omega^*(t) = \Pi_\omega(t) \in \mathcal{L}\big(\mathbf{V}_n^0(\Omega)\big) \quad \text{and} \quad \Pi_\omega(t) \geqslant 0, \\
&\text{for all } \mathbf{y} \in \mathbf{V}_n^0(\Omega), \quad \Pi_\omega(t)\mathbf{y} \in \mathbf{V}^2(\Omega) \cap \mathbf{V}_0^1(\Omega) \quad \text{and} \quad \big|\Pi_\omega(t)\mathbf{y}\big|_{\mathbf{V}^2(\Omega)} \leqslant C|\mathbf{y}|_{\mathbf{V}_n^0(\Omega)},
\end{aligned}$$

for $t \geqslant t_0$, $\Pi_\omega(t) = \widehat{\Pi}_\omega$, where $\widehat{\Pi}_\omega$ is the solution to the algebraic equation

$$\widehat{\Pi}_\omega = \widehat{\Pi}_\omega^* \geqslant 0, \quad A_\omega^*\widehat{\Pi}_\omega + \widehat{\Pi}_\omega A_\omega - \widehat{\Pi}_\omega BM^2B^*\widehat{\Pi}_\omega + (-A_0)^{-1} = 0, \tag{7.8}$$

for $t \leqslant t_0$, Π_ω is the solution to the differential equation

$$-\Pi_\omega'(t) = A_\omega^* \Pi_\omega + \Pi_\omega A_\omega - \theta^2(t)\Pi_\omega B M^2 B^* \Pi_\omega + (-A_0)^{-1},$$
$$\Pi_\omega(t_0) = \widehat{\Pi}_\omega.$$

The existence of a unique solution to this system may be proved as in Section 5. Consider the Navier–Stokes equations (7.7) with the linear feedback law $\mathbf{u}(t) = -\theta(t) M B^* \Pi_\omega(t) P\mathbf{y}(t)$:

$$\begin{aligned} &P\mathbf{y}' = A_{\omega,\Pi_\omega}(t)P\mathbf{y} - \mathrm{e}^{-\omega t} P\big((\mathbf{y}\cdot\nabla)\mathbf{y}\big) \quad \text{in } (0,\infty), \quad P\mathbf{y}(0) = \mathbf{y}_0,\\ &(I-P)\mathbf{y} = -(I-P)\theta^2 D_A M^2 B^* \Pi_\omega(t) P\mathbf{y} \quad \text{in } (0,\infty), \end{aligned} \tag{7.9}$$

where

$$A_{\omega,\Pi_\omega}(t) = A + \omega I - \theta^2(t) B M^2 B^* \Pi_\omega(t).$$

As previously, if $\hat{\mathbf{y}}$ is a solution to (7.9), then $\mathbf{y} = \mathrm{e}^{-\omega t}\hat{\mathbf{y}}$ is the solution of

$$\begin{aligned} &P\mathbf{y}' = AP\mathbf{y} - \theta^2 B M^2 B^* \Pi_\omega P\mathbf{y} + PF(\mathbf{y}), \quad P\mathbf{y}(0) = \mathbf{y}_0,\\ &(I-P)\mathbf{y} = -(I-P)\theta^2 D_A M^2 B_n^* \Pi_\omega P\mathbf{y}. \end{aligned} \tag{7.10}$$

Theorem 7.2. *For all $0 < \varepsilon \leqslant 1/2$, there exist $\mu_0 = \mu_0(\mathbf{w}, \varepsilon, \omega) > 0$ and $\bar{C}_0(\mathbf{w}, \varepsilon, \omega)$, such that if $\mu \in [0, \mu_0]$ and $|\mathbf{y}_0|_{\mathbf{V}_n^{1/2+\varepsilon}(\Omega)} \leqslant \bar{C}_0(\mathbf{w}, \varepsilon, \omega)\mu$, Eq.* (7.10) *admits a unique solution $\mathbf{y}$ such that $P\mathbf{y}$ belongs to D_μ, where*

$$D_\mu = \big\{\mathbf{y} \in \mathbf{V}^{3/2+\varepsilon, 3/4+\varepsilon/2}(Q_\infty) \;\big|\; \big\|\mathrm{e}^{\omega(\cdot)}\mathbf{y}\big\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \leqslant \mu\big\}.$$

Moreover $\mathbf{y}$, which belongs to $C_b([0,\infty); \mathbf{V}^{1/2+\varepsilon}(\Omega))$, satisfies

$$\big|\mathbf{y}(t)\big|_{\mathbf{V}^{1/2+\varepsilon}(\Omega)} \leqslant C(\mathbf{w}, \varepsilon, \omega)\mu\, \mathrm{e}^{-\omega t}.$$

Proof. The proof can be performed as in the two-dimensional case, see [19, Theorem 6.7]. □

Appendix A

In this section we prove some regularity results for the state and the adjoint equations.

Lemma A.1. *If $\mathbf{y}_0 \in \mathcal{V}^\sigma(\Omega)$ with $-2 \leqslant \sigma \leqslant 2$, then the weak solution to the equation*

$$\mathbf{y}' = (A - \lambda_0 I)\mathbf{y} \quad \textit{in } (0,\infty), \quad \mathbf{y}(0) = \mathbf{y}_0,$$

obeys

$$\|\mathbf{y}\|_{L^2(0,\infty;\mathcal{V}^{\sigma+1}(\Omega))} + \|\mathbf{y}\|_{H^1(0,\infty;\mathcal{V}^{\sigma-1}(\Omega))} \leqslant C|\mathbf{y}_0|_{\mathcal{V}^\sigma(\Omega)}.$$

In particular $\mathbf{y}$ belongs to $C_b([0,\infty); \mathcal{V}^\sigma(\Omega))$ and satisfies

$$\|\mathbf{y}\|_{C_b([0,\infty);\mathcal{V}^\sigma(\Omega))} \leqslant C|\mathbf{y}_0|_{\mathcal{V}^\sigma(\Omega)}.$$

In the case where $\sigma = 1/2 + \varepsilon$ and $0 < \varepsilon \leqslant 1/2$, it yields

$$\|\mathbf{y}\|_{C_b([0,\infty);\mathbf{V}_0^{1/2+\varepsilon}(\Omega))} + \|\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \leqslant C|\mathbf{y}_0|_{\mathbf{V}_0^{1/2+\varepsilon}(\Omega)}.$$

Proof. We are going to see that it is sufficient to combine estimates which are classical over a finite time interval (see e.g. [7, Chapter 3, Theorem 2.2]), together with the exponential stability of the semigroup $(\mathrm{e}^{t(A-\lambda_0 I)})_{t\geqslant 0}$.

Step 1. Let us first consider $\sigma = 0$. We can write

$$\frac{1}{2}\big|\mathbf{y}(t)\big|^2_{\mathbf{V}_n^0(\Omega)} + \int_0^t \int_\Omega \big|(\lambda_0 I - A)^{1/2}\mathbf{y}\big|^2 = \frac{1}{2}|\mathbf{y}_0|^2_{\mathbf{V}_n^0(\Omega)},$$

for all $t>0$, which gives

$$\|\mathbf{y}\|_{L^\infty(0,\infty;\mathbf{V}_n^0(\Omega))}+\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}_0^1(\Omega))}\leqslant C|\mathbf{y}_0|_{\mathbf{V}_n^0(\Omega)}.$$

With the equation satisfied by $\mathbf{y}$ we have

$$|\mathbf{y}|_{H^1(0,\infty;\mathcal{V}^{-1}(\Omega))}\leqslant C|\mathbf{y}|_{L^2(0,\infty;\mathbf{V}_0^1(\Omega))}.$$

Thus the proof of the lemma is complete in the case when $\sigma=0$.

Step 2. Assume now that $\mathbf{y}_0\in\mathcal{V}^\sigma(\Omega)$. In that case $(A-\lambda_0 I)^{-\sigma/2}\mathbf{y}_0\in\mathcal{V}^0(\Omega)=\mathbf{V}_n^0(\Omega)$, and $(A-\lambda_0 I)^{-\sigma/2}\mathbf{y}$ obeys the equation:

$$(A-\lambda_0 I)^{-\sigma/2}\mathbf{y}'=(A-\lambda_0 I)(A-\lambda_0 I)^{-\sigma/2}\mathbf{y}\quad\text{in }(0,\infty),$$
$$(A-\lambda_0 I)^{-\sigma/2}\mathbf{y}(0)=(A-\lambda_0 I)^{-\sigma/2}\mathbf{y}_0.$$

From step 1, we deduce that

$$\left|(A-\lambda_0 I)^{-\sigma/2}\mathbf{y}\right|_{L^2(0,\infty;\mathbf{V}_0^1(\Omega))}+\left|(A-\lambda_0 I)^{-\sigma/2}\mathbf{y}\right|_{H^1(0,\infty;\mathbf{V}^{-1}(\Omega))}\leqslant C\left|(A-\lambda_0 I)^{-\sigma/2}\mathbf{y}_0\right|_{\mathbf{V}_n^0(\Omega)}.$$

Therefore we have

$$|\mathbf{y}|_{L^2(0,\infty;\mathcal{V}^{\sigma+1}(\Omega))}+|\mathbf{y}|_{H^1(0,\infty;\mathcal{V}^{\sigma-1}(\Omega))}\leqslant C|\mathbf{y}_0|_{\mathcal{V}^\sigma(\Omega)}.$$

The proof is complete. □

Lemma A.2. *If* $\mathbf{f}\in L^1(0,\infty;\mathbf{V}_n^0(\Omega))$, *the solution* $\mathbf{y}$ *to the equation*

$$\mathbf{y}'=(A-\lambda_0 I)\mathbf{y}+\mathbf{f}\quad\textit{in }(0,\infty),\qquad \mathbf{y}(0)=0,$$

obeys

$$\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}^1(\Omega))}\leqslant C\|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))}.$$

If $0<\varepsilon\leqslant 1/2$ *and* $\mathbf{f}\in L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')$, *then*

$$\|\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)}\leqslant C\|\mathbf{f}\|_{L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')}.$$

Proof. *Step 1.* Let us first give a very short proof of the following estimate

$$\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}^{1-\varepsilon'}(\Omega))}\leqslant C\|\mathbf{f}\|_{L^1(0,\infty;\mathbf{V}_n^0(\Omega))}\quad\text{for all }\varepsilon'>0.\tag{A.1}$$

From the identity

$$(-A+\lambda_0 I)^{1/2-\varepsilon'/2}\mathbf{y}(t)=\int_0^t(-A+\lambda_0 I)^{1/2-\varepsilon'/2}\mathrm{e}^{(t-\tau)(A-\lambda_0 I)}\mathbf{f}(\tau)\,\mathrm{d}\tau,$$

it follows that

$$\left\|\mathbf{y}(t)\right\|_{\mathbf{V}^{1-\varepsilon'}(\Omega)}\leqslant C\int_0^t\mathrm{e}^{-\omega(t-\tau)}(t-\tau)^{-1/2+\varepsilon'/2}|\mathbf{f}(\tau)|_{\mathbf{V}_n^0(\Omega)}\,\mathrm{d}\tau.$$

Thus, with Young's inequality for convolutions (see [22, p. 32]), we can prove the above estimate. Actually we can take $\varepsilon'=0$ in estimate (A.1). For that it is sufficient to adapt the proof of [24, p. 179] to an infinite time interval. See also [23, Theorem 2.3.1, Chapter 4].

Step 2. Now we assume that $\mathbf{f}$ belongs to $L^2(0,\infty;(\mathbf{V}_n^{1/2-\varepsilon}(\Omega))')$. To prove the estimate in that case we proceed by interpolation. We know that

$$\|\mathbf{y}\|_{\mathbf{V}^{2,1}(Q_\infty)}\leqslant C\|\mathbf{f}\|_{L^2(0,\infty;\mathbf{V}_n^0(\Omega))}\quad\text{and}\quad\|\mathbf{y}\|_{\mathbf{V}^{1,1/2}(Q_\infty)}\leqslant C\|\mathbf{f}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}.$$

Since we have:

$$\begin{aligned}\big[L^2\big(0,\infty;\mathbf{V}^0_n(\Omega)\big),L^2\big(0,\infty;\mathcal{V}^{-1}(\Omega)\big)\big]_{1/2-\varepsilon}&=\big(\big[L^2\big(0,\infty;\mathbf{V}^0_n(\Omega)\big),L^2\big(0,\infty;\mathbf{V}^1_0(\Omega)\big)\big]_{1/2-\varepsilon}\big)'\\&=L^2\big(0,\infty;\big(\mathbf{V}^{1/2-\varepsilon}_n(\Omega)\big)'\big),\end{aligned}$$

we obtain the desired result by interpolation. □

Lemma A.3. *If* $\mathbf{u}$ *belongs to* $\mathbf{V}^{\sigma,\sigma/2}(\Sigma_\infty)$ *with* $0\leqslant\sigma<1$*, then the weak solution to the equation*

$$\mathbf{y}'=(A-\lambda_0 I)\mathbf{y}+BM\mathbf{u}\quad \text{in } (0,\infty),\qquad \mathbf{y}(0)=0,$$

obeys

$$\|\mathbf{y}\|_{\mathbf{V}^{1/2+\sigma-\varepsilon,1/4+\sigma/2-\varepsilon/2}(Q_\infty)}\leqslant C\|\mathbf{u}\|_{\mathbf{V}^{\sigma,\sigma/2}(\Sigma_\infty)}\quad \text{for all } \varepsilon>0.$$

If $\mathbf{u}$ *belongs to* $\mathbf{V}^{\sigma,\sigma/2}(\Sigma_\infty)$ *with* $1<\sigma\leqslant 2$*, and if* $\mathbf{u}(0)=0$*, then*

$$\|\mathbf{y}\|_{\mathbf{V}^{1/2+\sigma-\varepsilon,1/4+\sigma/2-\varepsilon/2}(Q_\infty)}\leqslant C\|\mathbf{u}\|_{\mathbf{V}^{\sigma,\sigma/2}(\Sigma_\infty)}\quad \text{for all } \varepsilon>0.$$

Proof. This result is already proved in [19, Lemma 7.3]. □

Remark A.1. The proof in [19] is based on the integral representation of solutions and Young's inequality for convolutions as used in [18, Proof of Theorem 2.3]. Using [18, Theorem 2.7], we can show that $\mathbf{y}$ can be estimated in $\mathbf{V}^{1/2+\sigma,1/4+\sigma/2}(Q_T)$ in function of $\|\mathbf{u}\|_{\mathbf{V}^{\sigma,\sigma/2}(\Sigma_T)}$ for all $0<T<\infty$. To know if we can take $T=\infty$, or similarly if we can take $\varepsilon=0$ in the estimates of Lemma A.3, is not immediate. Indeed, according to [16, Theorem 2.1], the constant C_T in the estimate

$$\|\mathbf{y}\|_{\mathbf{V}^{1/2+\sigma,1/4+\sigma/2}(Q_T)}\leqslant C_T\|\mathbf{u}\|_{\mathbf{V}^{\sigma,\sigma/2}(\Sigma_T)},$$

is a nondecreasing function of T. Here taking advantage of the exponential stability the semigroup, we can probably show that C_T can be chosen independent of T. But this is not yet proved.

Lemma A.4. *For all* $\mathbf{y}\in\mathbf{V}^2_n(\Omega)$*, the solution* $\Phi\in\mathbf{V}^2(\Omega)\cap\mathbf{V}^1_0(\Omega)$ *to the stationary equation* $\lambda_0\Phi-A^*\Phi=\mathbf{y}$ *obeys*

$$|\Phi|_{\mathbf{V}^4(\Omega)}\leqslant C|\mathbf{y}|_{\mathbf{V}^2_n(\Omega)}.$$

Proof. We rewrite the equation in the form

$$\lambda_0\Phi-\nu P\Delta\Phi=\mathbf{y}-P\big((\mathbf{w}\cdot\nabla)\Phi\big)+P\big((\nabla\mathbf{w})^T\Phi\big).$$

Since $\mathbf{w}\in\mathbf{V}^3(\Omega)$ and $\Phi\in\mathbf{V}^2(\Omega)$, then $P((\mathbf{w}\cdot\nabla)\Phi)$ and $P((\nabla\mathbf{w})^T\Phi)$ belong to $\mathbf{V}^1(\Omega)$, which gives an estimate of Φ in $\mathbf{V}^3(\Omega)$. Knowing that $\Phi\in\mathbf{V}^3(\Omega)$, $P((\mathbf{w}\cdot\nabla)\Phi)$ and $P((\nabla\mathbf{w})^T\Phi)$ belong to $\mathbf{V}^2(\Omega)$, and the proof is complete. □

Lemma A.5. *Let* α *be in* $[0,1/2]$*. If the function* $\mathbf{y}$ *belongs to* $\mathbf{V}^{\sigma,\sigma/2}(Q_\infty)\cap L^2(0,\infty;\mathbf{V}^0_n(\Omega))$ *with* $0\leqslant\sigma\leqslant 2$*, then the solution* Φ *to the equation*

$$-\Phi'=(A^*-\lambda_0 I)\Phi+(-A_0)^{-2\alpha}\mathbf{y}\quad \text{in } (0,\infty),\qquad \Phi(\infty)=0,\tag{A.2}$$

satisfies

$$\|\Phi\|_{L^2(0,\infty;\mathbf{V}^{2+\sigma+4\alpha}(\Omega))}+\|\Phi\|_{H^{1+\sigma/2}(0,\infty;\mathbf{V}^{4\alpha}(\Omega))}\leqslant C\|\mathbf{y}\|_{\mathbf{V}^{\sigma,\sigma/2}(Q_\infty)}.\tag{A.3}$$

If $\alpha=1/2$ *and if* $\mathbf{y}$ *belongs to* $L^2(0,\infty;\mathcal{V}^{-1}(\Omega))$*, then the solution* Φ *to Eq.* (A.2) *obeys*:

$$\|\Phi\|_{L^2(0,\infty;\mathbf{V}^3(\Omega))}+\|\Phi\|_{H^1(0,\infty;\mathbf{V}^1_0(\Omega))}\leqslant C\|\mathbf{y}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}.\tag{A.4}$$

Proof. *Step 1.* Estimate (A.3) is already proved in [19, Lemma 7.5] in the case when $\alpha=0$. Let us establish this estimate for $\sigma=0$ and $0\leqslant\alpha\leqslant 1/2$. It is clear that $(-A_0)^{-2\alpha}\mathbf{y}$ belongs to $L^2(0,\infty;\mathbf{V}^0_n(\Omega))$. Thus applying

[19, Lemma 7.6], we first prove that Φ belongs to $\mathbf{V}^{2,1}(Q_\infty)$ with the corresponding estimate. Observe that $\Phi = \Phi_1 + \Phi_2$, where Φ_1 is the solution to

$$-\Phi_1' = (\nu A_0 - \lambda_0 I)\Phi_1 + (-A_0)^{-2\alpha}\mathbf{y} \quad \text{in } (0,\infty), \quad \Phi_1(\infty) = 0,$$

and Φ_2 is the solution to

$$-\Phi_2' = (\nu A_0 - \lambda_0 I)\Phi_2 - P\big((\mathbf{w}\cdot\nabla)\Phi\big) + P\big((\nabla \mathbf{w})^T\Phi\big) \quad \text{in } (0,\infty), \quad \Phi_2(\infty) = 0.$$

To study the regularity of Φ_1, we set $\hat{\Phi} = (-A_0)^{2\alpha}\Phi_1$. Since

$$\Phi_1(t) = \int_t^\infty \mathrm{e}^{(\tau - t)(\nu A_0 - \lambda_0 I)}(-A_0)^{-2\alpha}\mathbf{y}(\tau)\,\mathrm{d}\tau,$$

then $\hat{\Phi}$ is defined by

$$\hat{\Phi}(t) = \int_t^\infty \mathrm{e}^{(\tau - t)(\nu A_0 - \lambda_0 I)}\mathbf{y}(\tau)\,\mathrm{d}\tau.$$

Thus $\hat{\Phi}$ belongs to $\mathbf{V}^{2,1}(Q_\infty)$, and Φ_1 belongs to $L^2(0,\infty;\mathbf{V}^{2+4\alpha}(\Omega)) \cap H^1(0,\infty;\mathbf{V}^{4\alpha}(\Omega))$ for all $0 \leqslant \alpha \leqslant 1/2$.

To study the regularity of Φ_2, we observe that $\Phi \in \mathbf{V}^{2,1}(Q_\infty)$ and $\mathbf{w} \in \mathbf{V}^3(\Omega)$. We can verify that $-P((\mathbf{w}\cdot\nabla)\Phi) + P((\nabla\mathbf{w})^T\Phi)$ belongs to $\mathbf{V}^{1,1/2}(Q_\infty)$. Thus applying [19, Lemma 7.6] we prove that $\Phi_2 \in \mathbf{V}^{3,3/2}(Q_\infty)$. If $\alpha \leqslant 1/4$, estimate (A.3) is proved for $\sigma = 0$.

Suppose that $\alpha \geqslant 1/4$. From the previous step we know that Φ belongs to $L^2(0,\infty;\mathbf{V}^3(\Omega)) \cap H^1(0,\infty;\mathbf{V}^1(\Omega))$. Hence $-P((\mathbf{w}\cdot\nabla)\Phi) + P((\nabla\mathbf{w})^T\Phi)$ belongs to $\mathbf{V}^{2,1}(Q_\infty)$. Thus Φ_2 belongs to $\mathbf{V}^{4,2}(Q_\infty)$, which proves estimate (A.3) for $\sigma = 0$ and $1/4 \leqslant \alpha \leqslant 1/2$.

Step 2. Let us first prove estimate (A.3) for $\sigma = 2$. As in step (i) we can show that $\hat{\Phi}$ belongs to $\mathbf{V}^{4,2}(Q_\infty)$, and Φ_1 belongs to $L^2(0,\infty;\mathbf{V}^{4+4\alpha}(\Omega)) \cap H^2(0,\infty;\mathbf{V}^{4\alpha}(\Omega))$. Thus $\Phi = \Phi_1 + \Phi_2$ belongs to $\mathbf{V}^{3,3/2}(Q_\infty)$. Since $\mathbf{w} \in \mathbf{V}^3(\Omega)$, we can verify that $-P((\mathbf{w}\cdot\nabla)\Phi) + P((\nabla\mathbf{w})^T\Phi)$ belongs to $\mathbf{V}^{2,1}(Q_\infty)$. Therefore Φ_2 belongs to $\mathbf{V}^{4,2}(Q_\infty)$, and $\Phi = \Phi_1 + \Phi_2$ belongs to $\mathbf{V}^{4,2}(Q_\infty)$. Now knowing that $\mathbf{w} \in \mathbf{V}^3(\Omega)$, we prove that $-P((\mathbf{w}\cdot\nabla)\Phi) + P((\nabla\mathbf{w})^T\Phi)$ belongs to $\mathbf{V}^{2+4\alpha,1+\alpha}(Q_\infty)$ provided that $2 + 4\alpha \leqslant 3$, i.e. $\alpha \leqslant 1/4$. Thus Φ_2 belongs to $\mathbf{V}^{4+4\alpha,2+\alpha}(Q_\infty)$ if $\alpha \leqslant 1/4$.

The case where $1/4 < \alpha \leqslant 1/2$ can be treated as in step (i). We know that Φ belongs to $L^2(0,\infty;\mathbf{V}^5(\Omega)) \cap H^2(0,\infty;\mathbf{V}^1(\Omega))$, and $-P((\mathbf{w}\cdot\nabla)\Phi) + P((\nabla\mathbf{w})^T\Phi)$ belongs to $L^2(0,\infty;\mathbf{V}^4(\Omega)) \cap H^2(0,\infty;\mathbf{V}^0(\Omega))$. Thus Φ_2 belongs to $\mathbf{V}^{6,3}(Q_\infty)$, which proves estimate (A.3) for $\sigma = 2$ and $\alpha \leqslant 1/2$.

Consequently estimate (A.3) is proved for $\sigma = 2$. The intermediate result can be proved by interpolation.

Step 3. Now, we consider the case where $\alpha = 1/2$ and $\mathbf{y} \in L^2(0,\infty;\mathcal{V}^{-1}(\Omega)) \cap H^1(0,\infty;\mathcal{V}^{-3}(\Omega))$. The function $(\lambda_0 I - A^*)^{1/2}\Phi$ obeys the equation

$$\begin{aligned}
&-\big(\lambda_0 I - A^*\big)^{1/2}\Phi' = \big(A^* - \lambda_0 I\big)\big(\lambda_0 I - A^*\big)^{1/2}\Phi + \big(\lambda_0 I - A^*\big)^{1/2}(-A_0)^{-1}\mathbf{y} \quad \text{in } (0,\infty),\\
&\big(\lambda_0 I - A^*\big)^{1/2}\Phi(\infty) = 0,
\end{aligned}$$

and

$$\big\|\big(\lambda_0 I - A^*\big)^{1/2}(-A_0)^{-1}\mathbf{y}\big\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))} \leqslant C\|\mathbf{y}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))}.$$

Therefore we have

$$\big\|\big(\lambda_0 I - A^*\big)^{1/2}\Phi\big\|_{\mathbf{V}^{2,1}(Q_\infty)} \leqslant C\|\mathbf{y}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))},$$

which yields to estimate (A.4). □

Lemma A.6. *Let α be in* $[0,1/2]$. *Let Φ be the solution to Eq.* (A.2)*, and let ψ be the pressure associated with Φ, that is to say the function ψ satisfying*

$$-\frac{\partial\Phi}{\partial t}-\nu\Delta\Phi-(\mathbf{w}\cdot\nabla)\Phi+(\nabla\Phi)^T\mathbf{w}+\lambda_0\Phi+\nabla\psi=(-A_0)^{-2\alpha}\mathbf{y}\quad\text{in } Q_\infty,$$
$$\operatorname{div}\Phi=0 \text{ in } Q_\infty,\quad \Phi=0 \text{ on } \Sigma_\infty,\quad \Phi(\infty)=0 \text{ in } \Omega. \tag{A.5}$$

If, in (A.2)*,* $\mathbf{y}$ *belongs to* $L^2(0,\infty;\mathbf{V}^0_n(\Omega))$*, then the function* ψ *belongs to* $L^2(0,\infty;H^{1+4\alpha}(\Omega))$*. If in addition* $\mathbf{y}$ *belongs to* $\mathbf{V}^{\sigma,\sigma/2}(Q_\infty)$ *with* $0\leqslant\sigma\leqslant 2$*, then the function* ψ *belongs to* $L^2(0,\infty;H^{\sigma+1+4\alpha}(\Omega))\cap H^{\sigma/2}(0,\infty;H^{1+4\alpha}(\Omega))$.

Proof. First observe that

$$\begin{aligned}\nabla\psi&=(I-P)\biggl((-A_0)^{-2\alpha}\mathbf{y}+\frac{\partial\Phi}{\partial t}+\nu\Delta\Phi+(\mathbf{w}\cdot\nabla)\Phi-(\nabla\Phi)^T\mathbf{w}\biggr)\\&=(I-P)\bigl(\nu\Delta\Phi+(\mathbf{w}\cdot\nabla)\Phi-(\nabla\Phi)^T\mathbf{w}\bigr).\end{aligned}$$

Assume that $\mathbf{y}$ belongs to $L^2(0,\infty;\mathbf{V}^\sigma_n(\Omega))\cap H^{\sigma/2}(0,\infty;\mathbf{V}^0_n(\Omega))$ with $0\leqslant\sigma\leqslant 2$. From Lemma A.5, it follows that

$$\bigl(\nu\Delta\Phi+(\mathbf{w}\cdot\nabla)\Phi-(\nabla\Phi)^T\mathbf{w}\bigr)\in L^2\bigl(0,\infty;\mathbf{H}^{\sigma+4\alpha}(\Omega)\bigr)\cap H^{\sigma/2}\bigl(0,\infty;\mathbf{H}^{4\alpha}(\Omega)\bigr).$$

Thus $\nabla\psi$ belongs to $L^2(0,\infty;\mathbf{V}^{\sigma+4\alpha}(\Omega))\cap H^{\sigma/2}(0,\infty;\mathbf{H}^{4\alpha}(\Omega))$, and the proof is complete. □

Lemma A.7. *Let* $\Phi\in\mathbf{V}^{2,1}(Q_\infty)$ *be the solution to Eq.* (A.2)*, and set*

$$\mathbf{u}=-\theta MB^*\Phi.$$

If, in (A.2)*, the function* $\mathbf{y}$ *belongs to* $L^2(0,\infty;\mathbf{V}^\sigma_n(\Omega))\cap H^{\sigma/2}(0,\infty;\mathbf{V}^0_n(\Omega))$ *with* $0\leqslant\sigma\leqslant 2$*, then*

$$\begin{aligned}&\|B^*\Phi\|_{L^2(0,\infty;\mathbf{V}^{\sigma+1/2+4\alpha}(\Gamma))\cap H^{\sigma/2}(0,\infty;\mathbf{V}^{1/2+4\alpha}(\Gamma))}+\|\mathbf{u}\|_{L^2(0,\infty;\mathbf{V}^{\sigma+1/2+4\alpha}(\Gamma))\cap H^{\sigma/2}(0,\infty;\mathbf{V}^{1/2+4\alpha}(\Gamma))}\\&\quad\leqslant C\|\mathbf{y}\|_{\mathbf{V}^{\sigma,\sigma/2}(Q_\infty)}.\end{aligned}\tag{A.6}$$

Proof. As in [19, Lemma 3.1] we can show that

$$\mathbf{u}=-\theta M\biggl(\nu\frac{\partial\Phi}{\partial\mathbf{n}}-\psi\mathbf{n}+c(\psi)\mathbf{n}\biggr),$$

where ψ is the pressure associated with Φ and $c(\psi)$ is the constant defined in (2.1).

Since $\mathbf{y}$ belongs to $L^2(0,\infty;\mathbf{V}^\sigma_n(\Omega))\cap H^{\sigma/2}(0,\infty;\mathbf{V}^0_n(\Omega))$, from Lemma A.5 we deduce that Φ belongs to $L^2(0,\infty;\mathbf{V}^{s+2+4\alpha}(\Omega))\cap H^{1+\sigma/2}(0,\infty;\mathbf{V}^{4\alpha}(\Omega))$, and from Lemma A.6 it follows that ψ belongs to $L^2(0,\infty;H^{\sigma+1+4\alpha}(\Omega))\cap H^{\sigma/2}(0,\infty;H^{1+4\alpha}(\Omega))$. Thus $\mathbf{u}$ belongs to $L^2(0,\infty;\mathbf{V}^{s+1/2+4\alpha}(\Gamma))\cap H^{\sigma/2}(0,\infty;\mathbf{V}^{1/2+4\alpha}(\Gamma))$. □

Lemma A.8. *Let* α *be in* $[0,1/2]$*. For all* $\mathbf{y}\in L^2(0,T;\mathbf{V}^0_n(\Omega))$*, the solution to the equation,*

$$-\Phi'=A^*\Phi+(-A_0)^{-2\alpha}\mathbf{y}\quad\text{in } (0,T),\quad \Phi(T)=0,\tag{A.7}$$

satisfies

$$\|\Phi\|_{L^2(0,T;\mathbf{V}^{(2+4\alpha)\wedge\eta}(\Omega))}+\|\Phi\|_{H^1(0,T;\mathbf{V}^{4\alpha\wedge(\eta/2-1/4)}(\Omega))}\leqslant C\|\mathbf{y}\|_{L^2(0,T;\mathbf{V}^0_n(\Omega))},\tag{A.8}$$

for all $\eta<7/2$*. If the function* $\mathbf{y}$ *belongs to* $\mathbf{V}^{\sigma,\sigma/2}(Q_T)$ *with* $0\leqslant\sigma<3/2$*, then the function* Φ *satisfies the following estimate holds*:

$$\|\Phi\|_{L^2(0,T;\mathbf{V}^{(\sigma+2+4\alpha)\wedge\eta}(\Omega))}+\|\Phi\|_{H^{s/2+1}(0,T;\mathbf{V}^{4\alpha\wedge(\eta/2-1/4-\sigma)}(\Omega))}\leqslant C\|\mathbf{y}\|_{\mathbf{V}^{\sigma,\sigma/2}(Q_T)}\tag{A.9}$$

for all $\eta<7/2$.

If $\alpha=1/2$ *and if* $\mathbf{y}$ *belongs to* $L^2(0,T;\mathcal{V}^{-1}(\Omega))$*, then the solution* Φ *to Eq.* (A.7) *obeys*:

$$\|\Phi\|_{L^2(0,T;\mathbf{V}^3(\Omega))}+\|\Phi\|_{H^1(0,T;\mathbf{V}^1_0(\Omega))}\leqslant C\|\mathbf{y}\|_{L^2(0,T;\mathcal{V}^{-1}(\Omega))}.\tag{A.10}$$

Proof. We proceed as in the proof of Lemma A.5.

Step 1. Let us establish estimate (A.8) for $0 < \alpha \leqslant 1/2$. It is clear that $(-A_0)^{-2\alpha}\mathbf{y}$ belongs to $L^2(0,\infty;\mathbf{V}^0_n(\Omega))$. Thus applying [19, Lemma 7.6], we first prove that Φ belongs to $\mathbf{V}^{2,1}(Q_T)$ with the corresponding estimate. As in the proof of Lemma A.5 we write $\Phi = \Phi_1 + \Phi_2$, where Φ_1 is the solution to

$$-\Phi_1' = (\nu A_0 - \lambda_0 I)\Phi_1 + (-A_0)^{-2\alpha}\mathbf{y} \quad \text{in } (0,T), \quad \Phi_1(T) = 0,$$

and Φ_2 is the solution to

$$-\Phi_2' = (\nu A_0 - \lambda_0 I)\Phi_2 - P\big((\mathbf{w}\cdot\nabla)\Phi\big) + P\big((\nabla\mathbf{w})^T\Phi\big) \quad \text{in } (0,T), \quad \Phi_2(T) = 0.$$

Since

$$\Phi_1(t) = \int_t^T \mathrm{e}^{(\tau-t)(\nu A_0-\lambda_0 I)}(-A_0)^{-2\alpha}\mathbf{y}(\tau)\,\mathrm{d}\tau,$$

then $\hat{\Phi} = (-A_0)^{2\alpha}\Phi_1$ is defined by

$$\hat{\Phi}(t) = \int_t^T \mathrm{e}^{(\tau-t)(\nu A_0-\lambda_0 I)}\mathbf{y}(\tau)\,\mathrm{d}\tau.$$

Thus $\hat{\Phi}$ belongs to $\mathbf{V}^{2,1}(Q_T)$, and Φ_1 belongs to $L^2(0,T;\mathbf{V}^{2+4\alpha}(\Omega)) \cap H^1(0,T;\mathbf{V}^{4\alpha}(\Omega))$ for all $0 \leqslant \alpha \leqslant 1/2$.

Let us study the regularity of Φ_2. Since $\Phi \in \mathbf{V}^{2,1}(Q_T)$ and $\mathbf{w} \in \mathbf{V}^3(\Omega)$, we can verify that $-P((\mathbf{w}\cdot\nabla)\Phi) + P((\nabla\mathbf{w})^T\Phi)$ belongs to $\mathbf{V}^{1,1/2}(Q_T)$. Thus applying [19, Lemma 7.6] we prove that $\Phi_2 \in \mathbf{V}^{3,3/2}(Q_T)$. Estimate (A.8) is proved for $\alpha \leqslant 1/4$.

Suppose that $\alpha \geqslant 1/4$. From the previous step we know that Φ belongs to $L^2(0,T;\mathbf{V}^3(\Omega)) \cap H^1(0,T;\mathbf{V}^1(\Omega))$. Hence $-P((\mathbf{w}\cdot\nabla)\Phi) + P((\nabla\mathbf{w})^T\Phi)$ belongs to $\mathbf{V}^{2,1}(Q_T)$. Thus Φ_2 belongs to $\mathbf{V}^{\eta,\eta/2}(Q_T)$ for all $\eta < 7/2$, which proves estimate (A.8) for $1/4 \leqslant \alpha \leqslant 1/2$.

Step 2. Now we assume that $\mathbf{y} \in \mathbf{V}^{2,1}(Q_T)$ and that $\mathbf{y}(T) \in \mathbf{V}^1_0(\Omega)$. As in step 1 we can show that $\hat{\Phi}$ belongs to $\mathbf{V}^{4,2}(Q_T)$, and Φ_1 belongs to $L^2(0,T;\mathbf{V}^{4+4\alpha}(\Omega)) \cap H^2(0,T;\mathbf{V}^{4\alpha}(\Omega))$. If $\mathbf{y} \in \mathbf{V}^{\sigma,\sigma/2}(Q_T)$ with $0 \leqslant \sigma < 3/2$, by interpolation between with the estimates obtained for $\sigma = 0$ and the one obtained for $\sigma = 2$, we obtain

$$\|\Phi_1\|_{L^2(0,T;\mathbf{V}^{\sigma+2+4\alpha}(\Omega))} + \|\Phi_1\|_{H^{\sigma/2+1}(0,T;\mathbf{V}^{4\alpha}(\Omega))} \leqslant C\|\mathbf{y}\|_{\mathbf{V}^{\sigma,\sigma/2}(Q_T)}.$$

For $\sigma < 3/2$ the regularity condition $y(T) \in \mathbf{V}^1_0(\Omega)$ is not needed.

Let us study the regularity of Φ_2. Since $\Phi \in \mathbf{V}^{2,1}(Q_T)$ and $\mathbf{w} \in \mathbf{V}^3(\Omega)$, we can verify that $-P((\mathbf{w}\cdot\nabla)\Phi) + P((\nabla\mathbf{w})^T\Phi)$ belongs to $\mathbf{V}^{1,1/2}(Q_T)$. Thus applying [19, Lemma 7.6] we prove that $\Phi_2 \in \mathbf{V}^{3,3/2}(Q_T)$. If $1 \leqslant \sigma < 3/2$, we know that $\Phi = \Phi_1 + \Phi_2$ belongs to $\mathbf{V}^{3,3/2}(Q_T)$ and that $\mathbf{w} \in \mathbf{V}^3(\Omega)$, we can verify that $-P((\mathbf{w}\cdot\nabla)\Phi) + P((\nabla\mathbf{w})^T\Phi)$ belongs to $\mathbf{V}^{2,1}(Q_T)$. Therefore Φ_2 belongs to $\mathbf{V}^{\eta,\eta/2}(Q_T)$ for all $\eta < 7/2$, and $\Phi = \Phi_1 + \Phi_2$ belongs to $\mathbf{V}^{\sigma,\sigma/2}(Q_T) + (L^2(0,T;\mathbf{V}^{\sigma+2+4\alpha}(\Omega)) \cap H^{\sigma/2+1}(0,T;\mathbf{V}^{4\alpha}(\Omega)))$. Estimate (A.9) is proved for $1 \leqslant \sigma < 3/2$. The estimate for $0 \leqslant \sigma \leqslant 1$ can be obtained by interpolation between the estimates obtained for $\sigma = 0$ and $\sigma = 1$.

Step 3. The proof of (A.10) is completely analogous to that of (A.4). □

Lemma A.9. *If in* (A.7) $\mathbf{y}$ *belongs to* $\mathbf{V}^{\sigma,\sigma/2}(Q_T)$ *with* $0 \leqslant \sigma < 3/2$, *then the function* ψ, *the pressure associated with* Φ, *belongs to* $L^2(0,T;H^{(\sigma+1+4\alpha)\wedge(\eta+1)}(\Omega)) \cap H^{\sigma/2}(0,T;H^{(1+4\alpha)\wedge(\eta/2+3/4-s)}(\Omega))$ *for all* $\eta < 3/2$.

Proof. It is sufficient to adapt the proof of Lemma A.6 and to use Lemma A.8 to obtain the desired result. □

Lemma A.10. *Let* $\mathbf{y}$ *be the weak solution to equation*

$$\mathbf{y}' = \big(A - \lambda_0(-A_0)^{-\alpha}\big)\mathbf{y} \quad \textit{in } (0,\infty), \quad \mathbf{y}(0) = \mathbf{y}_0.$$

If $\mathbf{y}_0 \in \mathcal{V}^{-2}(\Omega)$, *then*

$$\|\mathbf{y}\|_{L^2(0,\infty;\mathcal{V}^{-1}(\Omega))} + \|\mathbf{y}\|_{H^1(0,\infty;\mathcal{V}^{-3}(\Omega))} \leqslant C|\mathbf{y}_0|_{\mathcal{V}^{-2}(\Omega)}.$$

If $\mathbf{y}_0 \in \mathbf{V}^0_n(\Omega)$*, then*

$$\|\mathbf{y}\|_{L^2(0,\infty;\mathbf{V}^1_0(\Omega))} + \|\mathbf{y}\|_{H^1(0,\infty;\mathbf{V}^{-1}(\Omega))} \leqslant C|\mathbf{y}_0|_{\mathbf{V}^0_n(\Omega)}.$$

If $\mathbf{y}_0 \in \mathbf{V}^{1/2+\varepsilon}_0(\Omega)$ *with* $0 < \varepsilon \leqslant 1/2$*, the weak solution to equation*

$$\mathbf{y}' = \big(A - \lambda_0(-A_0)^{-\alpha}\big)\mathbf{y} \quad \textit{in } (0,\infty), \quad \mathbf{y}(0) = \mathbf{y}_0,$$

obeys

$$\|\mathbf{y}\|_{C_b([0,\infty);\mathbf{V}^{1/2+\varepsilon}_0(\Omega))} + \|\mathbf{y}\|_{\mathbf{V}^{3/2+\varepsilon,3/4+\varepsilon/2}(Q_\infty)} \leqslant C|\mathbf{y}_0|_{\mathbf{V}^{1/2+\varepsilon}_0(\Omega)}.$$

Proof. It is sufficient to combine estimates which are classical over a finite time interval (see e.g. [7, Chapter 3, Theorem 2.2]), together with the exponential stability of the semigroup $(e^{t(A-\lambda_0(-A_0)^{-\alpha})})_{t\geqslant 0}$. □

Lemma A.11. *If* $\mathbf{u}$ *belongs to* $\mathbf{V}^{\sigma,\sigma/2}(\Sigma_\infty)$ *with* $0 \leqslant \sigma < 1$*, then the weak solution to the equation*

$$\mathbf{y}' = \big(A - \lambda_0(-A_0)^{-\alpha}\big)\mathbf{y} + \theta BM\mathbf{u} \quad \textit{in } (0,\infty), \quad \mathbf{y}(0) = 0,$$

obeys

$$\|\mathbf{y}\|_{\mathbf{V}^{1/2+\sigma-\varepsilon,1/4+\sigma/2-\varepsilon/2}(Q_\infty)} \leqslant C\|\mathbf{u}\|_{\mathbf{V}^{\sigma,\sigma/2}(\Sigma_\infty)} \quad \textit{for all } \varepsilon > 0.$$

If $\mathbf{u}$ *belongs to* $\mathbf{V}^{\sigma,\sigma/2}(\Sigma_\infty)$ *with* $1 < \sigma \leqslant 2$*, and if* $\mathbf{u}(0) = 0$*, then*

$$\|\mathbf{y}\|_{\mathbf{V}^{1/2+s-\varepsilon,1/4+s/2-\varepsilon/2}(Q_\infty)} \leqslant C\|\mathbf{u}\|_{\mathbf{V}^{\sigma,\sigma/2}(\Sigma_\infty)} \quad \textit{for all } \varepsilon > 0.$$

Proof. We can follows the lines of the proof given in [19, Lemma 7.3] if we are able to show that, for all $0 < \varepsilon \leqslant 1/2$, $(\lambda_0(-A_0)^{-\alpha} - A)^{1/4-\varepsilon/2}PD_A$ is bounded from $\mathbf{V}^0(\Gamma)$ into $\mathbf{V}^0_n(\Omega)$. We know that $(\lambda_0 I - A)^{1/4-\varepsilon/2}PD_A$ is bounded from $\mathbf{V}^0(\Gamma)$ into $\mathbf{V}^0_n(\Omega)$. Thus it is enough to prove that $(\lambda_0(-A_0)^{-\alpha} - A)^{1/4-\varepsilon/2}(\lambda_0 I - A)^{-1/4+\varepsilon/2}$ is bounded in $\mathbf{V}^0_n(\Omega)$. It is the case since $(\lambda_0 I - A)^{-1/4+\varepsilon/2}$ is an isomorphism from $\mathbf{V}^0_n(\Omega)$ into $\mathbf{V}^{1/2-\varepsilon}_n(\Omega)$, and $(\lambda_0(-A_0)^{-\alpha} - A)^{1/4-\varepsilon/2}$ is an isomorphism from $\mathbf{V}^{1/2-\varepsilon}_n(\Omega)$ into $\mathbf{V}^0_n(\Omega)$. □

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Further reading

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